Math 30 Counting Problems

November 25, 2024

1. How many arrangements could be made of the word:

   - FATHER if F is first?

     $$\underline{F}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5! = 120$$

   - UNCLE if C is first and L is last?

     $$C\ \underbrace{\_\ \_\ \_}_{3! = 6}\ L$$

   - DAUGHTER if UG is last?

     $$\underline{\quad 6! \quad}\ \underline{GU}$$

     $$720$$

   - MOTHER if the vowels are first and last?

     $$2! \cdot 4! = 48$$

2. Determine the number of different arrangements of the 6 letter word ANSWER

   - Without restrictions

     $$6! = 720$$

   - That begin with an S

     $$\underline{S}\ \underline{\quad 5! \quad} = 120$$

   - That begin with a vowel and end with a consonant.

     $$\underbrace{\underline{2}}_{\text{vowels}} \cdot \underbrace{\underline{4!}}_{\text{rearrange the middle}} \cdot \underbrace{\underline{4}}_{\text{consonants}} = 192$$

   - That have the three letters ANS adjacent and in that order.

     $$\underbrace{\underline{ANS}\ \underline{W}\ \underline{E}\ \underline{R}}_{4! = 24}$$

   - That have the three letters ANS adjacent and in any order.

     $$\underbrace{\overset{3!}{\underline{ANS}}\ \underline{W}\ \underline{E}\ \underline{R}}_{4! \cdot 3! = 24 \cdot 6 = 144}$$

3. Eric, James, Lucas, Jayant, and Jovan foo to watch a movie and sit in 5 adjacent seats. In how how many ways can this be done if

- Eric sits next to Lucas?

$$\underbrace{\underbrace{E\ \ L}_{2!}\ \_\_\ \_\_\ \_\_}_{4!} = 2! \cdot 4! = 48$$

- Scott refuses to sit next to Jovan?

$$5! - 48 = 72$$

All ways ↑ (5!)

ways they are together ↑ (48)

4. In how many ways can four adults and five children be arranged in a single line

- Without restriction?

$a_1\ a_2\ a_3\ a_4\ c_1\ c_2\ c_3\ c_4\ c_5$

$$9! = 362880$$

- If the children and adults alternate positions?

$c_1\ a_1\ c_2\ a_2\ c_3\ a_3\ c_4\ a_4\ c_5$

Need child on either end: $5! \cdot 4! = 120 \cdot 24 = 2880$

- If the adults are all together and the children are all together?

$$5! \cdot 4! \cdot 2! = 5760$$

$$\underbrace{\overset{4!}{\text{adults}}\ \ \overset{5!}{\text{childs}}}_{2!}$$

- If the adults are all together?

$$\underbrace{\text{adults}\ \ c_1\ c_2\ c_3\ c_4\ c_5}_{}$$

$$6! \cdot 4! = 720 \cdot 24 = 17280$$

5. How many different arrangements can be made using all the letters of each word?

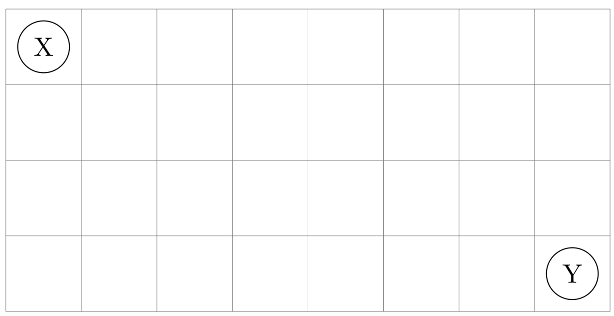
- RENERT $\dfrac{6!}{2!\,2!} = \dfrac{720}{4} = 180$

- ELLIANA $\dfrac{7!}{2!\,2!} = 1260$

- XOXXXXOOOXXXX

$$\frac{13!}{9!\cdot 4!} = \binom{13}{4} = 715$$

6. How many ways can you travel from X to Y if you may only travel to adjacent squares right or down?



$$\underbrace{rrrrrrrddd}$$

$$\frac{10!}{7!\,3!} = \frac{10\cdot 9\cdot 8}{6}$$

$$= 10\cdot 3\cdot 4$$

$$= 120$$

7. How many ways can you travel from X to Y if you may travel one or two units and only to squares right or down?

8 cases : 1) $\underbrace{r_1r_1r_1r_1r_1r_1r_1\,d_1d_1d_1}$

$$= \frac{10!}{7!\,3!} = \frac{10\cdot 9\cdot 8}{6} = 120$$

2) $\underbrace{r_1r_1r_1r_1r_1r_1r_1\,d_2d_1}$

$$\frac{9!}{7!} = 9\cdot 8 = 72$$

3) $\underbrace{r_2r_1r_1r_1r_1r_1\,d_1d_1d_1}$

$$\frac{9!}{5!\,3!} = \frac{9\cdot 8\cdot 7\cdot 6}{6} = 9\cdot 8\cdot 7 = 504$$

4) $\underbrace{r_2r_1r_1r_1r_1r_1\,d_2d_1}$

$$\frac{8!}{5!} = 8\cdot 7\cdot 6 = 336$$

5) $\underbrace{r_2r_2r_1r_1r_1\,d_1d_1d_1}$

$$\frac{8!}{2!\,3!\,3!}$$

6) $\underbrace{r_2r_2r_1r_1r_1\,d_2d_1}$

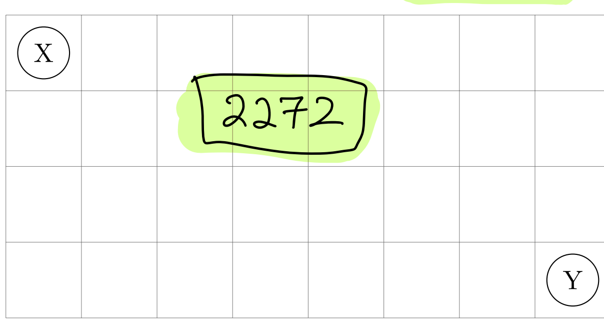
$$\frac{7!}{2!\,3!}$$

7) $\underbrace{r_2r_2r_2r_1\,d_1d_1d_1}$

$$\frac{7!}{3!\,3!}$$

8) $\underbrace{r_2r_2r_2r_1\,d_2d_1}$

$$\frac{6!}{3!}$$



8. • How many 5 card poker hands are possible?

$$\underbrace{\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48}}_{5!} = \binom{52}{5} = \boxed{259860}$$

• How many hands will there be all diamonds?

$$\underbrace{\underline{13}\cdot\underline{12}\cdot\underline{11}\cdot\underline{10}\cdot\underline{9}}_{5!} \qquad \binom{13}{5} = \boxed{1287}$$

• How many hands will there be 4 black cards and 1 red card?

$$\frac{26\cdot 25\cdot 24\cdot 23}{4!}\cdot\frac{\underline{26}}{1!} \qquad \binom{26}{4}\binom{26}{1} = \boxed{388\,700}$$

• How many hands will have 3 kings?

$$\binom{4}{3}\cdot\binom{48}{2} = \boxed{4512}$$

9. Jovan's pizza store has 9 choices of toppings available.

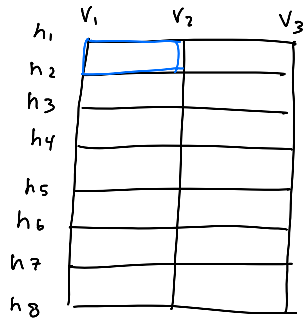
• How many different 2-topping pizzas can be made?

$$\binom{9}{2} = \boxed{36}$$

• How many different 3-topping pizzas can be made?

$$\binom{9}{3} = \boxed{84}$$

10. How many different rectangles can be formed by eight horizontal lines and three vertical lines?



$h_1\ h_2\ v_1\ v_2$

Take 2 horizontal and 2 vertical lines.

$$\binom{8}{2}\binom{3}{2} = \boxed{84}$$

11. A basketball coach has five guards and seven forwards on his basketball team.

- In how many different ways can he select a starting team of two guards and three forwards?

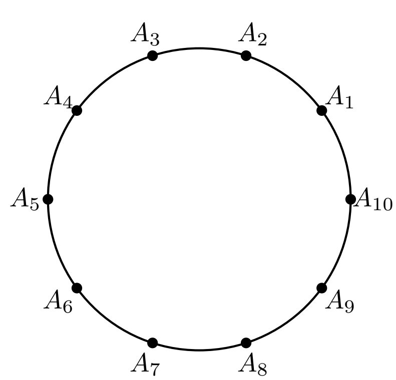
$g_1\, g_2\, g_3\, g_4\, g_5\quad f_1\, f_2\, f_3\, f_4\, f_5\, f_6\, f_7$

$$\binom{5}{2}\cdot\binom{7}{3} = \boxed{350}$$

- How many different starting teams are there if the star player, who plays guard, must be included?

$$(1)\cdot\binom{4}{1}\cdot\binom{7}{3} = \boxed{140}$$

12. How many chords can be formed between the points $A_1, A_2, \ldots, A_{10}$?



$$\binom{10}{2} = \boxed{45}$$

13. How many different 4 card hands have

- At least one black card?

Could have: 0, 1, 2, 3, 4

$$\underbrace{\binom{52}{4}}_{\text{All hands}} - \underbrace{\binom{26}{4}}_{\text{No Black}} = \boxed{255775}$$

- At least 2 kings?

0, 1, 2, 3, 4

Both ways have 3 terms

$$\binom{4}{2}\binom{48}{2} + \binom{4}{3}\binom{48}{1} + \binom{4}{4}\binom{48}{0} = \boxed{6961}$$

- Two pairs?

Rank	2	2	4	4
suit	♡	♠	♠	♡

$\binom{13}{2}$ ways to choose 2 'ranks'!

2 2 4 4
5 5 6 6

$\binom{4}{2}$ ways to choose first suit

$\binom{4}{2}$ ways to choose second suit

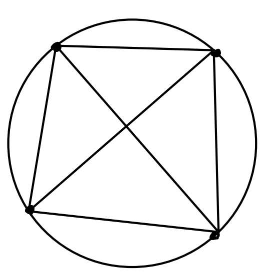
$$\binom{13}{2}\binom{4}{2}\binom{4}{2} = \boxed{2808}$$

* note order doesn't matter because of rank

- At most 2 clubs?

$$\binom{52}{4} - \binom{13}{3}\binom{39}{1} - \binom{13}{4}\binom{39}{0} = \boxed{28856}$$

14. Show that the number of diagonals in a p-sided polygon is $\dfrac{p(p-3)}{2}$



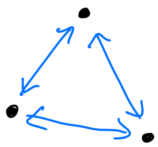
$$\underline{\#\text{ chords}} - \underline{\#\text{ sides}}$$

$$\binom{p}{2} - p = \frac{p\cdot(p-1)}{2} - p = \frac{1}{2}\left(p(p-1) - 2p\right)$$

$$= \frac{1}{2}(p^2 - p - 2p)$$

$$= \frac{1}{2}(p^2 - 3p) = \boxed{\frac{1}{2}p(p-3)}$$

15. After everyone had shaken hands once with everyone else in a room, there was a total of 66 handshakes. How many people were in the room?



$$\binom{n}{2} = 66$$

$$\frac{n(n-1)}{2} = 66$$

$$n^2 - n - 132 = 0$$

$$(n-12)(n+11) = 0$$

$$\boxed{n = 12}$$

16. Collinear points are points which share the same straight line. Find the number of triangles which can be formed from 10 points if no three of the points are collinear.

$$\binom{10}{3} = \frac{10\cdot 9\cdot 8}{6} = 10\cdot 3\cdot 4 = \boxed{120}$$

17. There are 5 different English books, 2 different Science books, and 2 different mathematics books.

- How many ways can three of these books be arranged on the shelf?

$$e_1\ e_2\ e_3\ e_4\ e_5 \quad s_1\ s_2 \quad m_1\ m_2$$

$$9\cdot 8\cdot 7 = 9\cdot 56 = \boxed{504}$$

- How many ways can two english, two science, and a math book be arranged?

$$\underbrace{\binom{5}{2}\cdot\binom{2}{2}\cdot\binom{2}{1}}_{\text{ways to select books}}\cdot 5! \;\Big\}\ \text{ways to arrange books of interest.}$$

$$\left(\frac{5\cdot 4}{2}\right)(2)(120) = 10\cdot 2(120) = \boxed{2400}$$

18. A coach must have 5 starters for a basketball team from 6 males and 5 females. If there must be at least two of each gender in the starting line-up, how many different groups of players can be chosen?

$m_1 m_2 m_3 m_4 m_5 m_6 \quad f_1 f_2 f_3 f_4 f_5$

2 cases

2 m 3 f -or- 3 m 2 f

$$\binom{6}{2}\binom{5}{3} + \binom{6}{3}\binom{5}{2} = \boxed{350}$$