Penguin Probability

Mr. Merrick · October 21, 2025

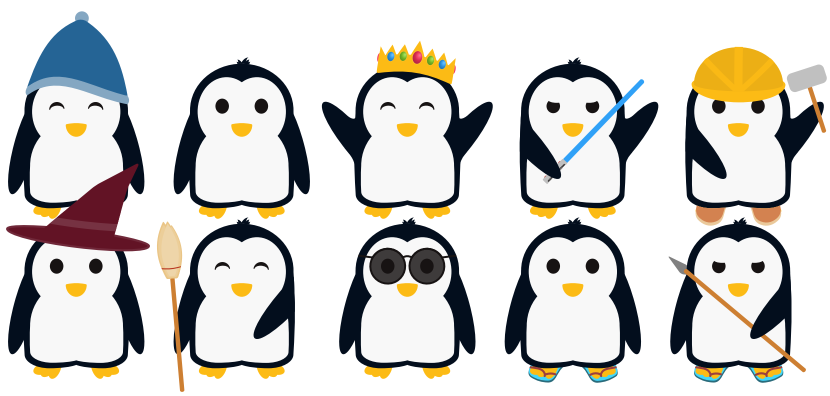
Explainer

Sets, Counting, and Probability

- A **set** is just a group of things. We'll use S for "the whole group."
- $|S|$ means "how many are in S." (This is called the **size** or **cardinality**.)
- An **event** is a smaller set inside S with a special property. Example: $H = \{\text{penguins that wear a hat}\}$.
- If we sample one penguin at random (all equally likely),

  $$P(\text{event}) = \frac{|\text{event}|}{|S|}.$$

  Think: "favorable penguins" over "all penguins."
- **Intersection** $A \cap B$ means "in A *and* in B." $P(A \cap B) = \dfrac{|A \cap B|}{|S|}$.

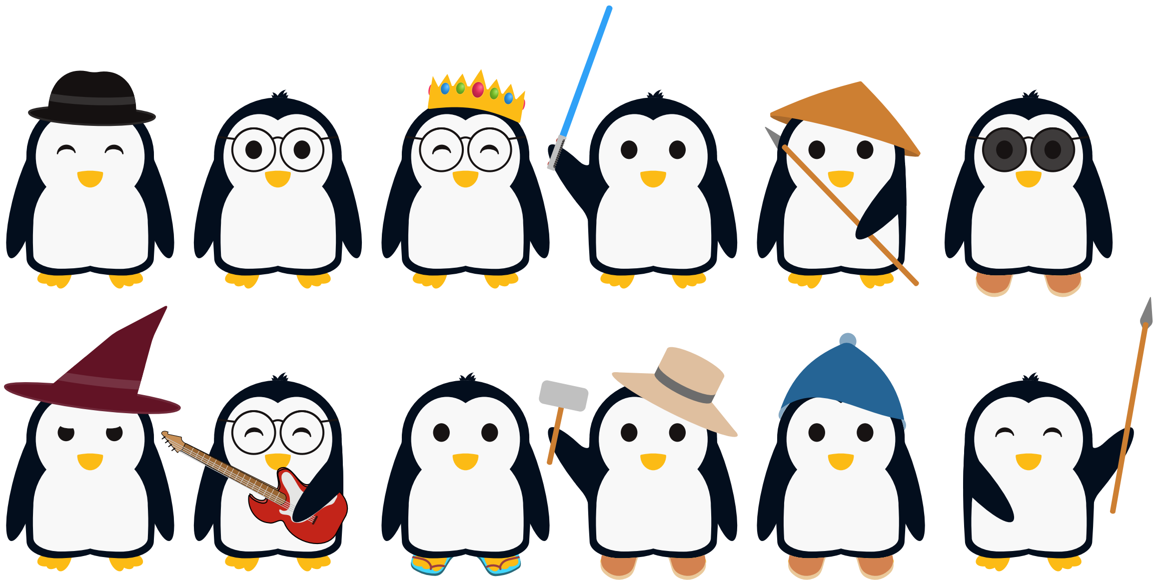


Let the set of 10 penguins above be S, and define three events:

$A = \{\text{wears a hat}\}$, $B = \{\text{holding at least one item}\}$, $C = \{\text{wears shoes (shoes or flip-flops)}\}$.

Suppose you sample a single penguin at random:

1. What are $|S|$, $|A|$, $|B|$, $|C|$?
2. Find $P(A)$, $P(B)$, $P(C)$.
3. Find $|A \cap B|$ and $P(A \cap B)$.
4. Find $|B \cap C|$ and $P(B \cap C)$.
5. Compute $|A \cap B \cap C|$ and $P(A \cap B \cap C)$.



Let the set of 12 penguins above be S, and define the events:

$A = \{\text{wears a hat}\}$, $B = \{\text{holding at least one item}\}$, $C = \{\text{wears footwear (shoes or flip-flops)}\}$.

1. What are $|S|$, $|A|$, $|B|$, $|C|$?

2. Find $P(A)$, $P(B)$, $P(C)$.

3. Compute $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and the corresponding probabilities.

4. Find $|A \cap B \cap C|$ and $P(A \cap B \cap C)$.

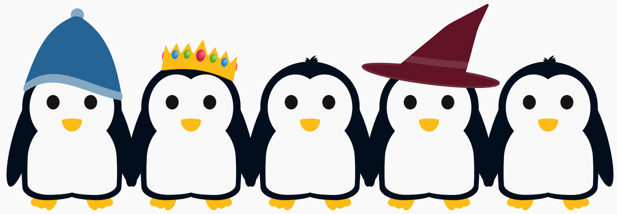
5. You are told that two *new* events A' and B' on this picture satisfy

$$|A'| = 4, \qquad |B'| = 4, \qquad |A' \cap B'| = 2.$$

   Give one possible, sensible *rule* for A' and one for B' that match these counts, using visible features only.

Explainer

What's Conditional Probability?
Imagine you sample one of five friendly penguins in the row below at random. Some are wearing hats, and one of those hats is a witch hat!

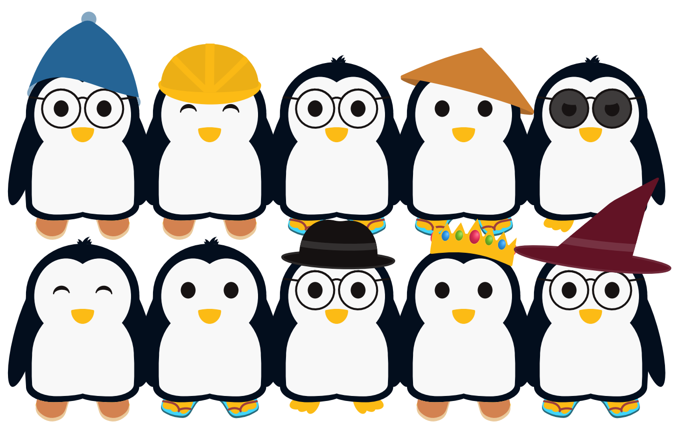


Out of 5 penguins:

- 3 have hats (wool, crown, or witch). $P(\text{hat}) = \frac{3}{5}$.
- Of the hatted penguins, 1 has a witch hat. $P(\text{witch hat} \mid \text{hat}) = \frac{1}{3}$.

Conditional probability means: first look only at the "hat world," then ask, "what fraction of these hats are witch hats?"

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}.$$



Consider the 10 penguins above. Let three sets be defined as follows:

$$G = \{\text{wearing glasses or sunglasses}\}, \quad W = \{\text{wearing footwear}\}, \quad H = \{\text{wearing any hat}\}.$$

If a penguin is selected at random determine each of the following:

1. $P(W)$
2. $P(W \cap H)$
3. $P(G \mid W)$
4. $P(H \mid W)$

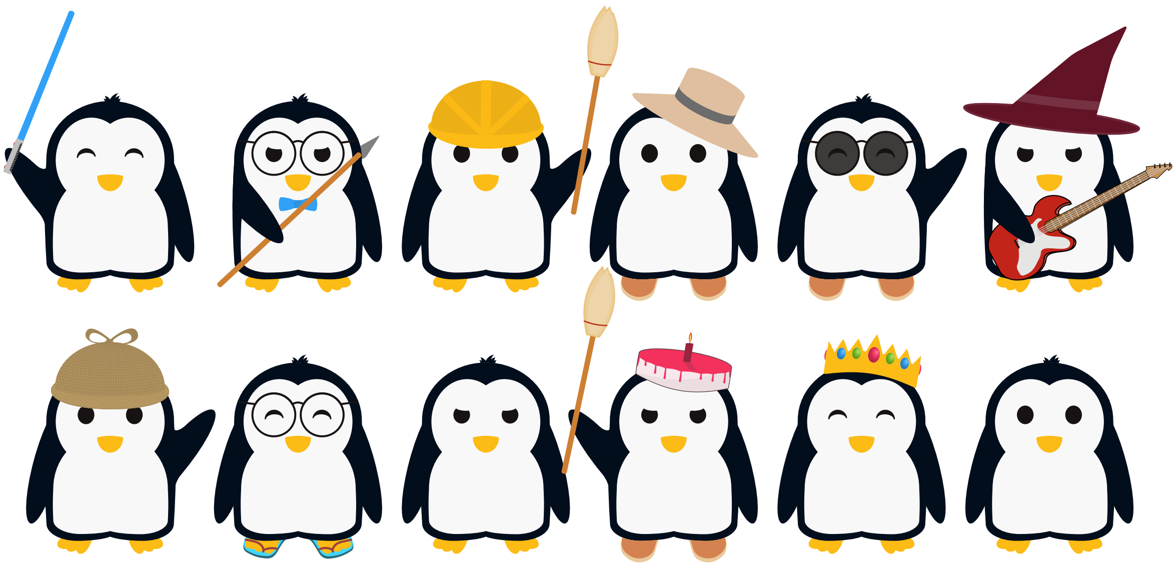
Consider the 24 penguins above. Let $H = \{\text{wearing any hat}\}$, $L = \{\text{holding an item in the left wing}\}$, $R = \{\text{holding an item in the right wing}\}$, $S = \{\text{wearing shoes}\}$, $F = \{\text{wearing flip-flops}\}$.

(a) Find $P(H)$, $P(L)$, $P(R)$, $P(S)$, $P(F)$ by counting.

(b) Compute $P(H \cap L)$ and $P(L \mid H)$.

(c) Compute $P(\text{wool hat} \mid H)$ and $P(F \mid S)$.

(d) Are H and R independent? Compare $P(R)$ and $P(R \mid H)$.



Consider the 12 penguins above. Each belongs to one or more of three hidden "rule sets," A, B, and C.

$$|A| = 6, \quad |B| = 5, \quad |C| = 4$$
$$|A \cap B| = 3, \quad |A \cap C| = 2, \quad |B \cap C| = 1$$
$$|A \cap B \cap C| = 1$$

(a) How many penguins belong to none of the rules?

(b) Infer one possible set of visual "rules" that fits the table exactly (It may be helpful to draw a Venn diagram to visualize intersectional counts).

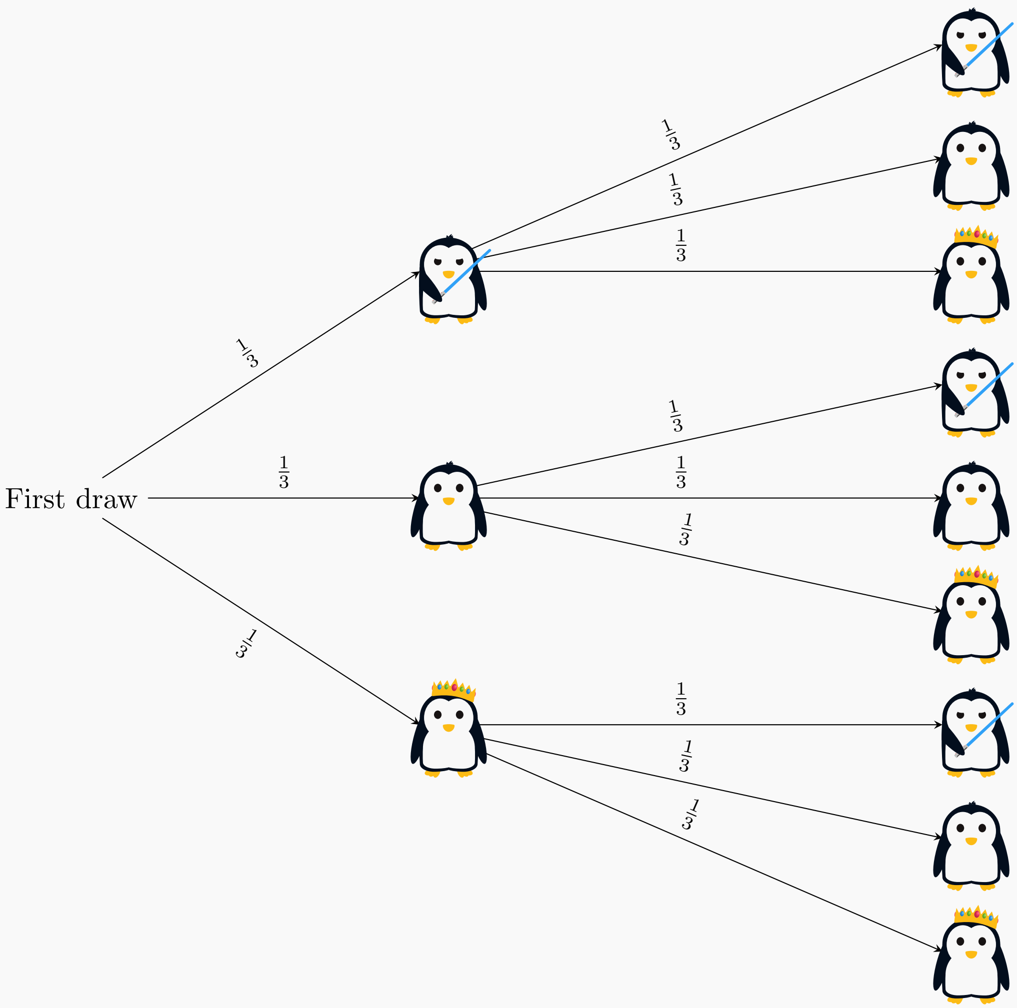
Explainer

Sampling *with replacement* (independent).
Consider the set of 3 penguins below:

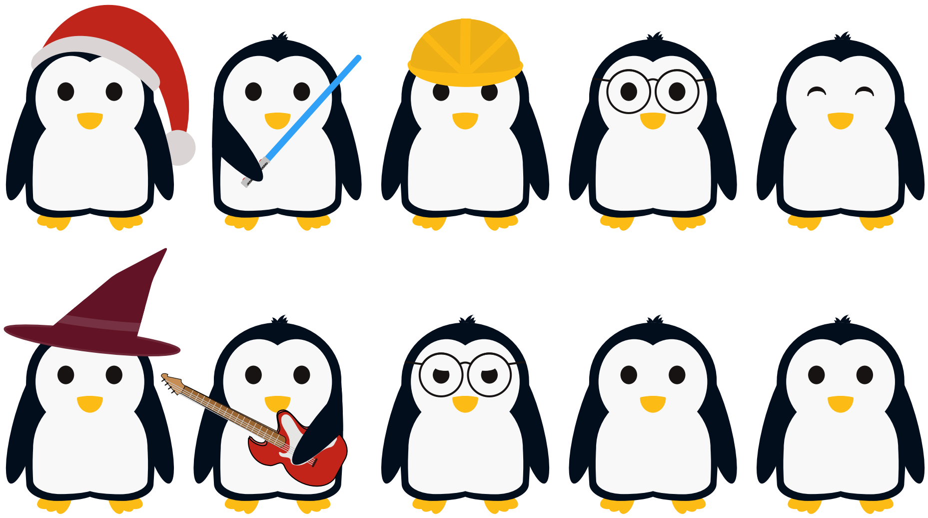
Sample one penguin, put it back into the set, and then sample again. This process is called sampling *with replacement*. Because the first penguin is replaced before the next draw, each penguin still has the same chance of being chosen ($\frac{1}{3}$). The two draws are therefore *independent*.

Example: Sampling the penguin with a lightsaber twice in a row:

$$P(L \text{ then } L) = P(L) \times P(L) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.$$



Out of the 9 total outcomes, one corresponds to drawing the lightsaber penguin twice in a row.



Let the set of 10 penguins above be S. Sample two penguins at random *with replacement* from S. Because the sampling is *with replacement*, each draw is independent and the probabilities stay the same from the first draw to the second. *Note: there are 3 hats and 7 without hats.*

1. Find $P(\text{hat})$ and $P(\text{no hat})$.

2. Find $P(\text{first has a hat, then second without a hat})$.

3. Find $P(\text{both have hats})$.

4. Find $P(\text{none have hats})$.

5. Find $P(\text{exactly one has a hat})$.

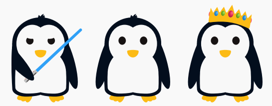
6. Find $P($*at least* one has a hat$)$.

7. Check that all possibilities (0 hats, 1 hat, or 2 hats) together make a total probability of 1.

Number of Hats	0	1	2
Probability			

Explainer

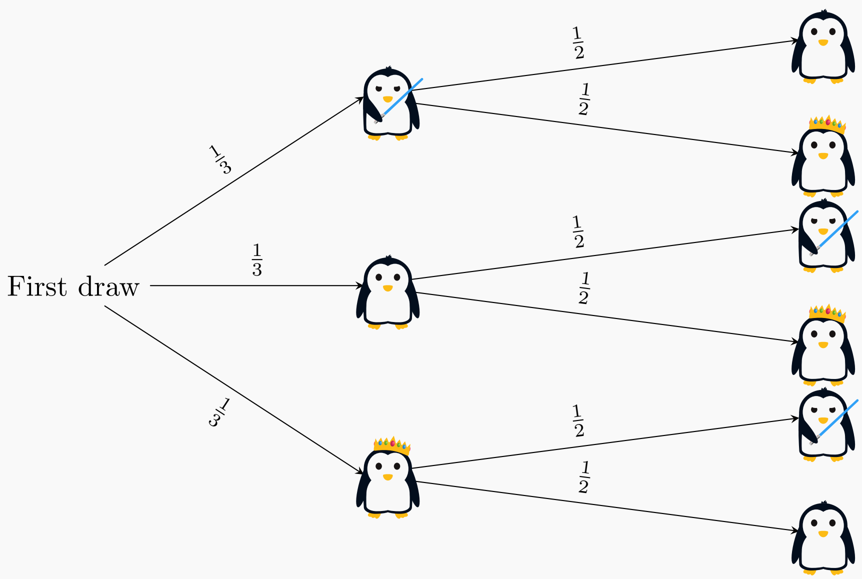
Sampling *without replacement* (dependent).
Consider the set of 3 penguins below:



Draw one penguin and *do not* put it back, then draw again. Because one penguin is removed from the set, the chances on the second draw *change*. The two draws are therefore *dependent*.
Example: "L then L", drawing the penguin with a lightsaber twice, is impossible now (you can't pick the same penguin twice), so

$$P(L \text{ then } L) = 0.$$

If the first draw is L, the second draw can only be P or C, each with probability $\frac{1}{2}$.

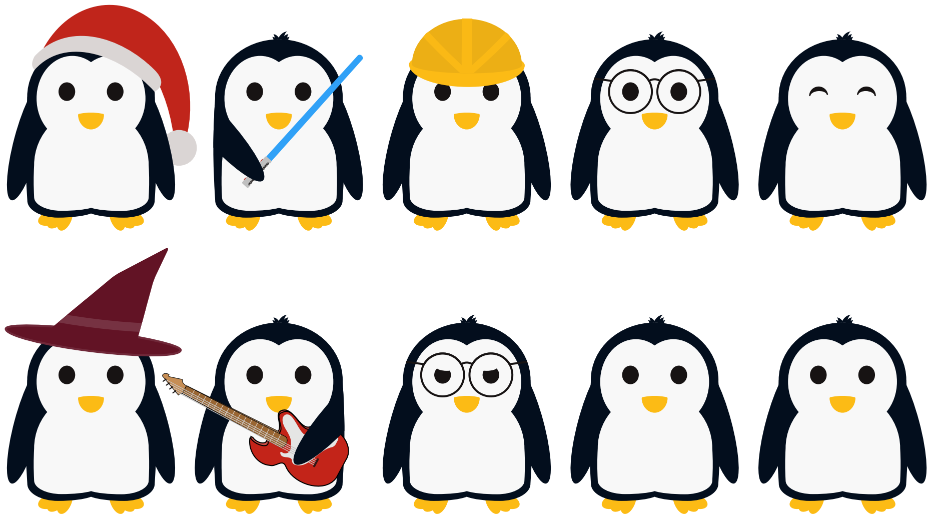


There are $3 \times 2 = 6$ equally likely outcomes. The second step has only two options because the first penguin is not replaced.

Draw two penguins at random *without replacement* from the set of 5 above.

1. Find P(first has glasses, then second does not).

2. Find P(*at least* one has glasses).



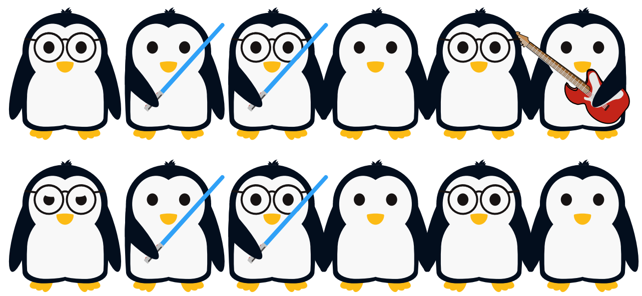
Draw two penguins *without replacement* at random from the set of 10 above. Because the first penguin is not returned to the set, the second-draw probabilities change.

1. Find P(hat on the first draw) and P(no hat on the first draw).
2. Find P(first has a hat, then second without a hat).
3. Find P(both have hats).
4. Find P(none have hats).
5. Find P(exactly one has a hat).
6. Find P(*at least* one has a hat).
7. Check that all possibilities (0 hats, 1 hat, or 2 hats) together make a total probability of 1.

Number of Hats	0	1	2
Probability			

Contingency Tables

The set below shows 12 penguins, each possibly wearing glasses, holding a lightsaber, both, or neither.



Each penguin can be described by two categorical variables: **Glasses (Yes/No)** and **Lightsaber (Yes/No)**. The contingency table below shows how many penguins fall into each category. Start by filling in the missing values.

	Lightsaber	**No Lightsaber**	
Glasses		4	
No Glasses			
			12

1. Assume one of the 12 penguins is sampled at random. Determine each of the following probabilities.
   (a) $P(\text{Glasses})$ and $P(\text{Lightsaber})$.

   (b) $P(\text{Glasses} \cap \text{Lightsaber})$.

   (c) $P(\text{Glasses} \mid \text{Lightsaber})$.

   (d) $P(\text{Lightsaber} \mid \text{Glasses})$.

2. Are "Glasses" and "Lightsaber" independent?

Contingency Table — Penguin Snack Choices

Three colonies of penguins (220 penguins total) pick their favorite snack: **Fish Pops (F)**, **Krill Cones (K)**, or **Snowcones (S)**. Start by filling in the missing values in the table below:

	Frostbite Fjord	Pebble Beach	Waddle Wharf	
Fish Pops (F)		15	20	
Krill Cones (K)	25			80
Snowcones (S)	10	25	20	55
	85		75	220

Suppose a single penguin is sampled at random from the 220 penguins. Determine each of the following:

(a) $P(\text{Frostbite Fjord} \cap F)$ and $P(F)$.

(b) $P(F \mid \text{Frostbite Fjord})$ and $P(\text{Frostbite Fjord} \mid F)$.

(c) $P(S \mid \text{Waddle Wharf})$.

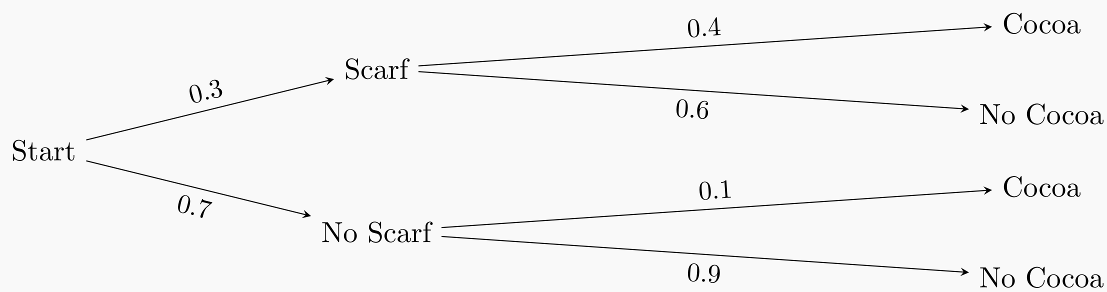
(d) Are "Colony" and "Snack" independent?

Tree Diagrams

Explainer

Tree diagrams help organize situations where one event affects another. Each branch represents a possible outcome, and by multiplying along a path, we find the probability of that specific sequence. Then, we can add across branches to find total probabilities.

Example: Penguins, Scarves, and Cocoa.
On cold mornings, there is a 30% chance a penguin wears a scarf. Given a penguin wears a scarf, there is a 40% chance they also carry cocoa. Among penguins without scarves, there is a 10% chance they carry cocoa. A tree diagram helps keep these conditional probabilities organized:



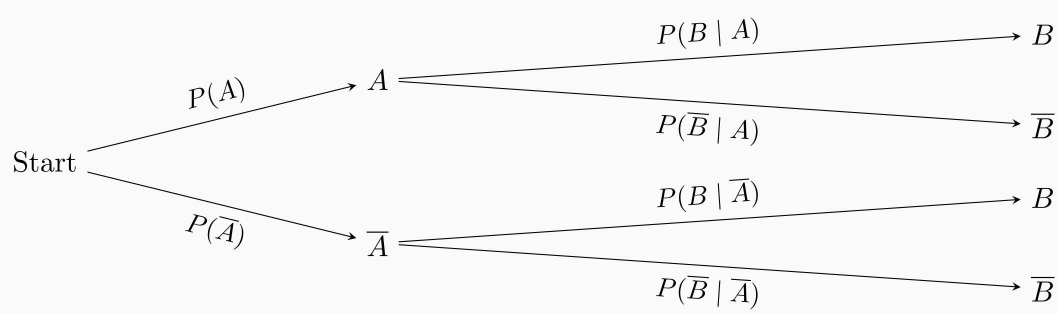
Multiply along each path to find joint probabilities:

$$P(\text{Scarf and Cocoa}) = 0.3 \times 0.4 = 0.12,$$
$$P(\text{No Scarf and Cocoa}) = 0.7 \times 0.1 = 0.07,$$
$$P(\text{Cocoa}) = 0.12 + 0.07 = 0.19.$$

Interpretation: About 19% of all penguins carry cocoa on a cold morning.

Heuristic: Out of 100 penguins, about 30 wear scarves and 70 do not. Of the 30 scarf-wearers, 40% (12) also carry cocoa. Of the 70 without scarves, 10% (7) carry cocoa. Altogether, about 19 penguins carry cocoa — matching $P(\text{Cocoa}) = 0.19$. Tree diagrams make these numbers easy to visualize.

General form.



Multiply along branches to find joint probabilities:

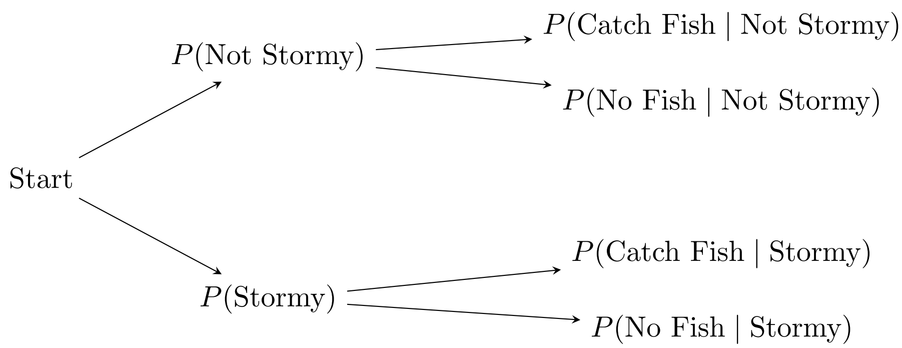
$$P(A \cap B) = P(A)\,P(B \mid A), \qquad P(\overline{A} \cap B) = P(\overline{A})\,P(B \mid \overline{A}),$$

and add them to find the overall probability of B.

Tree Diagram — Penguin Fishing Success

On the icy coast, penguins plan their daily fishing trips. There is a 60% chance it is stormy on a given day. When it is not stormy, penguins catch fish 80% of the time. When it is stormy, they catch fish 25% of the time. Suppose a day is chosen at random.

(a) Fill in the tree diagram below with the correct probabilities for each branch.



(b) Find the overall probability that a penguin catches fish.

(c) Find the probability that it was stormy and no fish were caught.

(d) If a penguin caught fish, what is the probability that it was not stormy?

Tree Diagram — Penguin Evening Activities

On chilly evenings, each penguin picks one activity to enjoy. There is a 30% chance a penguin joins the Snowball Fight, a 25% chance it joins the Ice Sculpting group, and a 45% chance it heads out on the Fishing Trip. Afterward, some penguins stop for a snow cone before heading home. If a penguin chose the Snowball Fight, there is a 35% chance they get a snow cone. If they chose Ice Sculpting, there is a 25% chance. And if they went on the Fishing Trip, there is a 15% chance they stop for one.

(a) Draw a tree diagram showing all possible outcomes and label each branch with its probability.

(b) Find the overall probability that a penguin gets a snow cone.

(c) If a penguin got a snow cone, what is the probability it had been in the Snowball Fight group?

(d) Find the probability that a penguin on the Fishing Trip does not get a snow cone.