Sampling Distribution of $\hat{p}$

AP Statistics · Mr. Merrick · January 11, 2026

Suppose we take a random sample of size n from a population.

- Let X be the number of successes in the sample.
- Let p be the true population proportion of successes.
- For a finite population of size N with K successes, $p = K/N$.
- The sample proportion is
$$\hat{p} = \frac{X}{n}.$$

Because the sample is random, X (and therefore $\hat{p}$) is a **random variable**. To make probability statements about $\hat{p}$, we must choose an appropriate probability model based on:

- whether observations can be treated as **independent**, and
- whether the **success–failure (normality) condition** is satisfied.

Row meaning: whether the **normality (success–failure) condition** is satisfied.
Column meaning: whether the **independence condition** is satisfied.

	Independence OK *(with replacement or $n \le 0.10N$)*	**Independence NOT OK** *(without replacement and $n > 0.10N$)*
Normality NOT OK *(success–failure fails)*	**Binomial (exact)** $X \sim \text{Bin}(n, p)$ $\hat{p} = X/n$ Use exact binomial probabilities.	**Hypergeometric (exact)** $X \sim \text{Hypergeometric}(N, K, n)$ $\hat{p} = X/n$ Use exact hypergeometric probabilities.
Normality OK *(success–failure holds)*	**Normal model for $\hat{p}$ (approx)** If independence is reasonable and $np \ge 10$, $n(1-p) \ge 10$, then $$\hat{p} \approx \mathcal{N}\left(p,\ \sqrt{\frac{p(1-p)}{n}}\right).$$ Use z-scores.	**Normal approximation (finite population)** When success–failure is met but the 10% condition fails, $$\hat{p} \approx \mathcal{N}\left(p,\ \sqrt{\frac{p(1-p)}{n}}\sqrt{\frac{N-n}{N-1}}\right)$$ $$p = \frac{K}{N}.$$

1. For each situation below, identify the **most appropriate probability model** for the sampling distribution of $\hat{p}$. *No probability calculations are required. Justify your choice.*

   (a) A public-opinion researcher randomly surveys $n = 10$ adults from a very large country. Historically, about 4% of adults support a particular third-party candidate.

   (b) A national polling organization surveys $n = 500$ randomly selected voters to estimate the proportion who approve of a proposed federal law. The true approval rate is 51%.

   (c) A teacher randomly selects $n = 18$ students *without replacement* from a class of $N = 30$ students to check who completed a summer assignment. Exactly $K = 12$ students in the class completed the assignment.

   (d) A basketball player practices by shooting $n = 7$ free throws. Based on past data, the probability of making a free throw is 0.85, and shots are assumed independent.

   (e) A shipment of $N = 60$ electronic components contains $K = 8$ defective items. A quality-control inspector randomly selects $n = 12$ components without replacement.

   (f) A town has $N = 500$ residents. A random sample of $n = 80$ residents is selected to estimate the proportion who own at least one dog. The true proportion is 0.40.

2. A school counselor believes that 65% of students participate in at least one club. A random sample of $n = 120$ students is selected from a school with 1,800 students.

   (a) Identify the appropriate probability model for the sampling distribution of $\hat{p}$.

   (b) Find the probability that the sample proportion of students who participate in at least one club is *greater than* 0.70.

3. A box contains $N = 90$ lightbulbs, $K = 15$ of which are defective. A technician randomly inspects $n = 20$ bulbs without replacement.

   (a) Explain why the normal model for $\hat{p}$ is *not* appropriate.

   (b) Find the probability that *at most 3* of the inspected bulbs are defective.

4. A news organization claims that 48% of adults in a certain state support a proposed ballot initiative. The organization randomly surveys $n = 400$ adults from the state.

   (a) Explain why the normal model for $\hat{p}$ is appropriate.

   (b) Find the probability that the sample proportion exceeds 0.52.

   (c) Interpret the result.

5. A manufacturing company claims that 30% of the items it produces fail an initial quality inspection. A random sample of $n = 150$ items is selected. Determine the probability that the sample proportion of failures is *less than* 0.25. Justify your model choice.