# Babylonian Method

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**The Babylonian Method**

$$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right), \qquad a > 0,\ x_0 > 0.$$

This is the Babylonian/Heron update for approximating $\sqrt{a}$.

**Tiny history.** The method goes back to ancient Babylon and appears in Heron's *Metrica* (1st century AD). It is the special case of Newton's method for $f(x) = x^2 - a$.

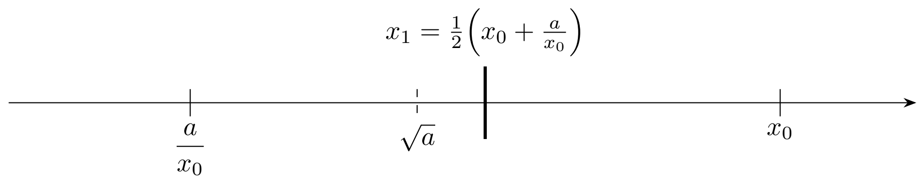
### How it Works

Core identity:

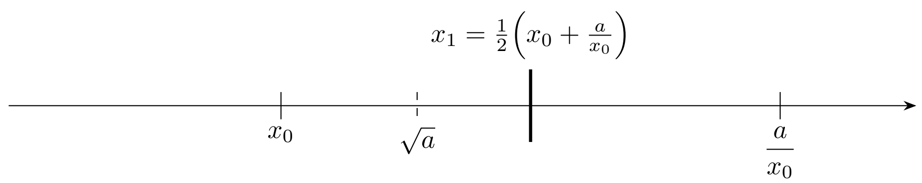
$$\sqrt{a}\left(\frac{\sqrt{a}}{x_0}\right) = \frac{a}{x_0}.$$

Hence $\sqrt{a}$ always lies strictly between $x_0$ and $\dfrac{a}{x_0}$. The update $x_1$ is their midpoint, so it moves closer to $\sqrt{a}$ regardless of whether $x_0$ starts above or below.

**Case 1:** $x_0 > \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} < 1 \Rightarrow \frac{a}{x_0} < \sqrt{a} < x_0$)




**Case 2:** $0 < x_0 < \sqrt{a}$ ($\frac{\sqrt{a}}{x_0} > 1 \Rightarrow x_0 < \sqrt{a} < \frac{a}{x_0}$)




## Example: Approximating $\sqrt{10}$

**Start above:** $x_0 = 4$

$x_1 = \frac{1}{2}\left(4 + \frac{10}{4}\right) = 3.25,$

$x_2 = \frac{1}{2}\left(3.25 + \frac{10}{3.25}\right) = 3.1634615\ldots,$

$x_3 = \frac{1}{2}\left(3.1634615\ldots + \frac{10}{3.1634615\ldots}\right) \approx 3.1622779.$

**Start below:** $x_0 = 3$

$x_1 = \frac{1}{2}\left(3 + \frac{10}{3}\right) = \frac{19}{6} \approx 3.1666667,$

$x_2 = \frac{1}{2}\left(\frac{19}{6} + \frac{10}{19/6}\right) \approx 3.1622807,$

$x_3 = \frac{1}{2}\left(3.1622807 + \frac{10}{3.1622807}\right) \approx 3.16227766.$

$$\boxed{\sqrt{10} \approx 3.16227766}$$

**Tip (digit–stability).** If two consecutive iterates agree to $k$ decimal places (i.e., $\mathrm{round}_k(x_n) = \mathrm{round}_k(x_{n+1})$), then those $k$ decimals are correct for $\sqrt{a}$.

## Practice: 2-Decimal Approximations

*Directions.* For each item, approximate $\sqrt{a}$ to **2 decimal places** starting from the given $x_0$ using

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Compute $x_1$ and $x_2$. If they already agree to 2 decimals, you're done (digit–stability tip).

1. $a = 2, \quad x_0 = 1$

$$\begin{aligned} x_1 &= \tfrac{1}{2}\big(1 + \tfrac{2}{1}\big) = \tfrac{3}{2} = 1.500000, \\ x_2 &= \tfrac{1}{2}\big(1.5 + \tfrac{2}{1.5}\big) = \tfrac{1}{2}(1.5 + 1.\overline{3}) = 1.416666\ldots, \\ \sqrt{2} &\approx \mathbf{1.41} \text{ (to 2 d.p.; } x_1 \text{ and } x_2 \text{ agree to 2 d.p.).} \end{aligned}$$

2. $a = 3, \quad x_0 = 1.5$

$$\begin{aligned} x_1 &= \tfrac{1}{2}\big(1.5 + \tfrac{3}{1.5}\big) = \tfrac{1}{2}(1.5 + 2) = 1.75, \\ x_2 &= \tfrac{1}{2}\big(1.75 + \tfrac{3}{1.75}\big) = \tfrac{1}{2}(1.75 + 1.714285\ldots) = 1.732142\ldots, \\ \sqrt{3} &\approx \mathbf{1.73} \text{ (to 2 d.p.).} \end{aligned}$$

3. $a = 5, \quad x_0 = 2$

$$\begin{aligned} x_1 &= \tfrac{1}{2}\big(2 + \tfrac{5}{2}\big) = \tfrac{1}{2}(2 + 2.5) = 2.25, \\ x_2 &= \tfrac{1}{2}\big(2.25 + \tfrac{5}{2.25}\big) = \tfrac{1}{2}(2.25 + 2.222222\ldots) = 2.236111\ldots, \\ \sqrt{5} &\approx \mathbf{2.24} \text{ (to 2 d.p.).} \end{aligned}$$

4. $a = 12, \quad x_0 = 3.5$

$$\begin{aligned} x_1 &= \tfrac{1}{2}\big(3.5 + \tfrac{12}{3.5}\big) = \tfrac{1}{2}(3.5 + 3.428571\ldots) = 3.464285\ldots, \\ x_2 &= \tfrac{1}{2}\big(3.464285\ldots + \tfrac{12}{3.464285\ldots}\big) \approx \tfrac{1}{2}(3.4642857 + 3.4639189) = 3.464102\ldots, \\ \sqrt{12} &\approx \mathbf{3.46} \text{ (to 2 d.p.).} \end{aligned}$$

5. $a = 50, \quad x_0 = 7$

$$\begin{aligned} x_1 &= \tfrac{1}{2}\big(7 + \tfrac{50}{7}\big) = \tfrac{1}{2}(7 + 7.142857\ldots) = 7.071428\ldots, \\ x_2 &= \tfrac{1}{2}\big(7.071428\ldots + \tfrac{50}{7.071428\ldots}\big) \approx \tfrac{1}{2}(7.0714286 + 7.0710679) = 7.071248\ldots, \\ \sqrt{50} &\approx \mathbf{7.07} \text{ (to 2 d.p.).} \end{aligned}$$