Countability and Uncountability Proofs

Mr. Merrick · Math 10 · September 16, 2025

Notation. $\mathbb{N} = \{1, 2, 3, \dots\}$ are the natural numbers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the reals. To say two sets can be put in "one-to-one correspondence" means we can pair each element of one set with exactly one element of the other, with nothing left over on either side.

The rational numbers $\mathbb{Q}$ can be put in one-to-one correspondence with $\mathbb{N}$

Theorem 1. *The rationals $\mathbb{Q}$ are countable: they can be listed in order so that each rational appears exactly once.*

Proof. Imagine a grid of fractions:

$$\begin{array}{cccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

We walk through the grid diagonally: first $\frac{1}{1}$, then $\frac{1}{2}, \frac{2}{1}$, then $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$, and so on. Each positive rational number appears somewhere in this grid. To avoid repeats, we only keep fractions in *lowest terms* (so $\frac{2}{2}$ is skipped, since it equals 1 which already appeared).
This gives a list $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \dots$ of all positive rationals. By weaving in their negatives and 0, we get a complete list of all rationals. Thus $\mathbb{Q}$ can be put in one-to-one correspondence with $\mathbb{N}$. □

The real numbers $\mathbb{R}$ cannot be put in one-to-one correspondence with $\mathbb{N}$

Theorem 2 (Cantor's diagonal argument)**.** *The real numbers in the interval $(0, 1)$ cannot be listed in a sequence. Therefore $\mathbb{R}$ is uncountable.*

Proof. Assume (for contradiction) that we can list all real numbers in $(0, 1)$ as $x_1, x_2, x_3, \dots$. Write each number in decimal form:

$$\begin{aligned} x_1 &= 0.\, a_{11}a_{12}a_{13}\dots \\ x_2 &= 0.\, a_{21}a_{22}a_{23}\dots \\ x_3 &= 0.\, a_{31}a_{32}a_{33}\dots \\ &\vdots \end{aligned}$$

Here $a_{ij} \in \{0, 1, 2, \dots, 9\}$ is the j-th digit of x_i after the decimal point. (If a number has two decimal expansions, like $0.4999\dots = 0.5$, pick the one that does *not* end with repeating 9s.) For clarity, picture the digits in a grid; row i holds the digits of x_i:

	1	2	3	$\cdots$
x_1	a_{11}	a_{12}	a_{13}	$\cdots$
x_2	a_{21}	a_{22}	a_{23}	$\cdots$
x_3	a_{31}	a_{32}	a_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now build a *new* number y by changing the digits along the diagonal of this grid. Define the j-th digit of y by

$$b_j = \begin{cases} 1, & \text{if } a_{jj} \neq 1, \\ 2, & \text{if } a_{jj} = 1. \end{cases}$$

Let $y = 0.\, b_1 b_2 b_3 \dots$. Then y *differs* from x_1 in the first digit, from x_2 in the second digit, from x_3 in the third digit, and so on. Therefore y is not equal to *any* number in the list $x_1, x_2, x_3, \dots$.

But we supposedly started with a complete list of all numbers in $(0, 1)$. Since y is a real number in $(0, 1)$ and is missing from the list, no such complete list can exist. Hence $(0, 1)$—and so $\mathbb{R}$—cannot be put in one-to-one correspondence with $\mathbb{N}$. □

Unions of countable and uncountable sets

Theorem 3. *The union of two countable sets is countable.*

Proof. Let A and B be countable sets. Then we can list their elements as

$$A = \{a_1, a_2, a_3, \dots\}, \quad B = \{b_1, b_2, b_3, \dots\}.$$

To form a list of $A \cup B$, we interleave the two lists:

$$a_1, b_1, a_2, b_2, a_3, b_3, \dots$$

This process gives a sequence containing all elements of $A \cup B$ (with possible repetitions, which can simply be skipped when they occur). Thus $A \cup B$ can be put in one-to-one correspondence with a subset of $\mathbb{N}$, and hence is countable. □

Theorem 4. *The union of a countable set and an uncountable set is uncountable.*

Proof. Let A be countable and B uncountable. Suppose for contradiction that $A \cup B$ is countable. Then both A and B would be subsets of a countable set, hence countable themselves. But this contradicts the assumption that B is uncountable. Therefore $A \cup B$ must be uncountable. □

The irrationals are "larger" than the rationals

Theorem 5. *The set of irrational numbers is uncountable, and therefore strictly larger in size than the rationals.*

Proof. We know $\mathbb{R}$ is uncountable, and $\mathbb{Q}$ is countable. If the irrationals (which are $\mathbb{R} \setminus \mathbb{Q}$) were also countable, then $\mathbb{R}$ would be a union of two countable sets, hence countable. This contradicts Cantor's diagonal argument. Therefore the irrationals are uncountable, while the rationals are countable. □

Remark. In plain words: there are "more" irrational numbers than rational numbers. Even though there are infinitely many rationals, they can be listed one-by-one. The irrationals cannot.

Vocabulary. In set theory, the symbol $\aleph_0$ (aleph-null) is used to denote the size of any countable infinity. For example, $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$. A set is called *countable* if it has size $\aleph_0$, meaning its elements can be listed one-by-one. The real numbers $\mathbb{R}$, and the irrationals $\mathbb{R} \setminus \mathbb{Q}$, are *uncountable*: they have size strictly larger than $\aleph_0$. The cardinality of the continuum (all real numbers) is usually written $\mathfrak{c} = 2^{\aleph_0}$. Thus:

- One-to-one correspondence $\leftrightarrow$ same size.
- Countable $\leftrightarrow$ size $= \aleph_0$ (listable).
- Uncountable $\leftrightarrow$ size $> \aleph_0$.
- $|\mathbb{Q}| = \aleph_0$, but $|\mathbb{R}| = |\text{irrationals}| = \mathfrak{c} > \aleph_0$.