Last Digits, and Trailing Zeros

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What is a Congruence?

A congruence is a way of saying two numbers have the same remainder when divided by some modulus. We write:

$$a \equiv b \pmod{m}$$

This means that a and b leave the same remainder when divided by m.

Example: $17 \equiv 2 \pmod{5}$, since both leave remainder 2 when divided by 5.

Why Congruences are Useful

Congruences let us shrink huge numbers down to their remainders. Instead of tracking a giant number like 7^{2025}, we only care about its remainder modulo some base.

Rules of the Game

Congruences behave similarly to equations, but with some important differences:

Operation	**Equations**	**Congruences**
Addition	$a = b \implies a + c = b + c$	$a \equiv b \pmod{m} \implies a + c \equiv b + c \pmod{m}$
Multiplication	$a = b \implies ac = bc$	$a \equiv b \pmod{m} \implies ac \equiv bc \pmod{m}$
Exponentiation	$a = b \implies a^k = b^k$	$a \equiv b \pmod{m} \implies a^k \equiv b^k \pmod{m}$
Division	$a = b \implies \frac{a}{c} = \frac{b}{c}$ (if $c \neq 0$)	**Not always valid!** Only allowed if c has a multiplicative inverse modulo m.

Cycles

When taking powers modulo some number, the results eventually repeat in cycles. This is the key idea behind many "last digit" problems.

Example: Powers of $7 \pmod{10}$

$$7,\ 9,\ 3,\ 1,\ 7,\ 9,\ 3,\ 1, \ldots$$

The cycle length is 4. To compute $7^{2025} \bmod 10$, we only need to know where 2025 lands in the cycle:

$$2025 \div 4 = 506 \text{ remainder } 1.$$

So 7^{2025} has the same last digit as 7^1, which is 7. Therefore the last digit is 7.

Systems of Congruences (Chinese Remainder Theorem)

Sometimes we want to solve problems with more than one modulus. This leads to a system of congruences, for example:

$$\begin{aligned} x &\equiv 2 \pmod{3}, \\ x &\equiv 3 \pmod{5}. \end{aligned}$$

Step 1: Write the possibilities for the first congruence. All numbers $\equiv 2 \pmod{3}$ are

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ \ldots \quad \text{(add 3 each time)}.$$

Step 2: Check which of these satisfy the second congruence. We need $x \equiv 3 \pmod{5}$. Among the list, 8 works, then $23, 38, \ldots$ (add 15 each time). So the full solution is $x \equiv 8 \pmod{15}$.

This example shows what the *Chinese Remainder Theorem* guarantees: - A solution exists when the moduli are coprime (3 and 5 are). - That solution is unique modulo the product of the moduli ($3 \cdot 5 = 15$).

Practice Problems

Try these on your own. Solutions appear if the macro is turned on.

1. Find the last digit of 3^{2024}.

   Solution: The cycle length is 4. $2024 \equiv 0 \pmod{4}$, so the last digit is 1.

2. Solve the system:

   $$\begin{aligned} x &\equiv 1 \pmod{4}, \\ x &\equiv 2 \pmod{5}. \end{aligned}$$

   Solution: From $x \equiv 1 \pmod{4}$ we get $1, 5, 9, 13, 17, \ldots$. The first that is $2 \pmod{5}$ is 17. So the solution is $x \equiv \mathbf{17} \pmod{20}$.

3. Solve the system:

   $$\begin{aligned} x &\equiv 2 \pmod{7}, \\ x &\equiv 3 \pmod{11}. \end{aligned}$$

   Solution: List $2, 9, 16, 23, 30, 37, 44, 51, 58, \ldots$ (numbers $\equiv 2 \pmod{7}$). Among these, $58 \equiv 3 \pmod{11}$. So the solution is $x \equiv \mathbf{58} \pmod{77}$.

4. Solve the system:

   $$\begin{aligned} x &\equiv 1 \pmod{3}, \\ x &\equiv 2 \pmod{4}, \\ x &\equiv 3 \pmod{5}. \end{aligned}$$

   Solution: From $x \equiv 1 \pmod{3}$ we get $1, 4, 7, 10, 13, \ldots$. Checking $2 \pmod{4}$ narrows to $10, 22, 34, 46, \ldots$. Among these, $58 \equiv 3 \pmod{5}$. So the solution is $x \equiv 58 \pmod{60}$.

Trailing Zeros in Big Factorials

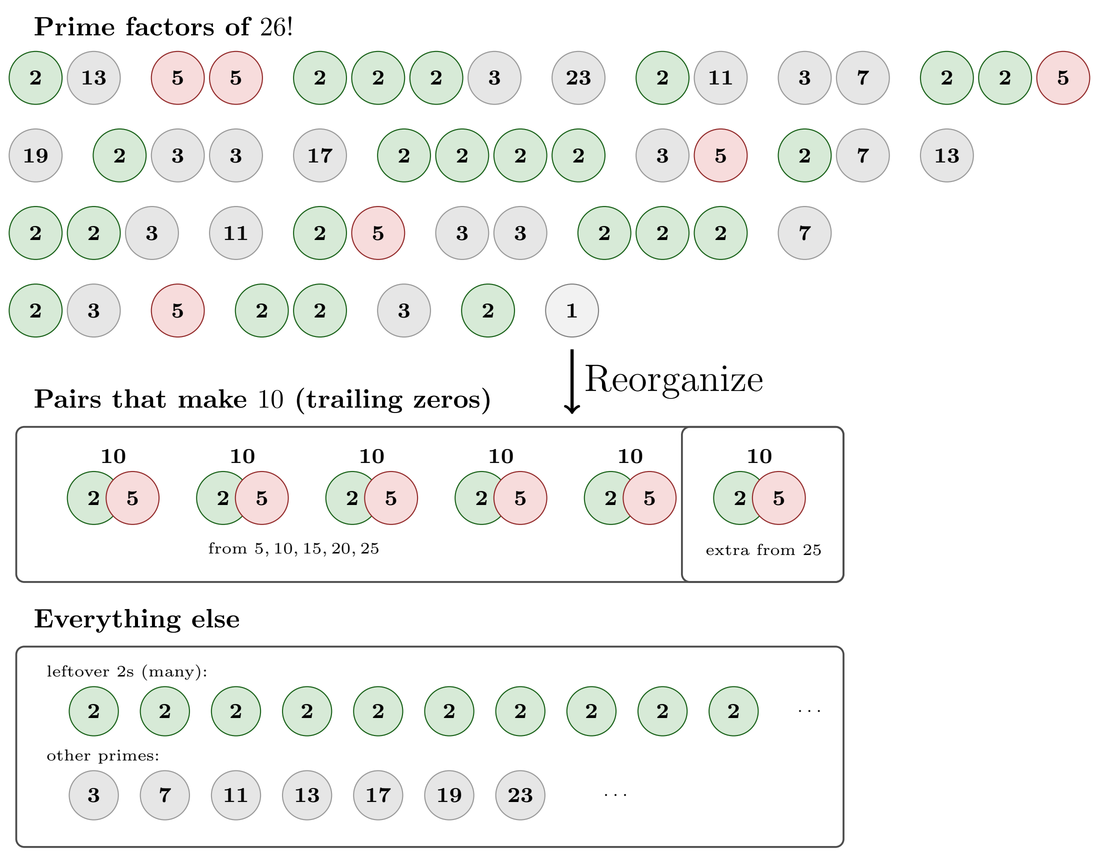
When we want to know how many zeros are at the end of 26!, we are really exploring how many times 10 divides 26!. Since $10 = 2 \cdot 5$, this comes down to counting how many pairs of $(2, 5)$ we can make inside the prime factorization of 26!. Notice that 26! has *lots* of factors of 2 (from all the even numbers), but only a limited number of factors of 5 (from multiples of 5, 10, 15, 20, 25, etc.). That means the 5's are the "bottleneck." Every $(2, 5)$ pair makes a trailing zero, so the number of zeros equals the number of 5's we can pull out.

This is just like a chemistry idea called the **limiting reagent**:

- If you want to make peanut butter sandwiches, you need both bread slices and spoonfuls of peanut butter. If you have 100 slices of bread but only 6 spoonfuls of peanut butter, you can only make 6 sandwiches.
- In making water ($H_2 + \frac{1}{2}O_2 \rightarrow H_2O$), the number of water molecules is limited by whichever ingredient — hydrogen or oxygen — runs out first.

In factorials, the 2's are like the bread: they are everywhere and we'll never run out. The 5's are like the peanut butter: they are much rarer, so they control how many "sandwiches" (i.e. tens) we can build.

Below is the factorization of 26!, reorganized to show the $(2, 5)$ pairs that make zeros:



5 pairs from multiples of 5 + 1 extra pair from 25 $\Rightarrow 5 + 1 = \mathbf{6}$ trailing zeros in 26!.

From this picture you can see: - 5 pairs come from the multiples of 5 $(5, 10, 15, 20, 25)$. - 1 extra pair comes from the extra factor of 5 inside $25 = 5 \cdot 5$.

That makes $5 + 1 = 6$ trailing zeros in 26!.

More Practice: Trailing Zeros

1. Count the zeros in 2025!.

   Solution: $v_5(2025!) = \lfloor\frac{2025}{5}\rfloor + \lfloor\frac{2025}{25}\rfloor + \lfloor\frac{2025}{125}\rfloor + \lfloor\frac{2025}{625}\rfloor = 405 + 81 + 16 + 3 = \mathbf{505}$. Hence 2025! ends with **505** zeros. (For reference, $v_2(2025!) = 2017 \gg 505$.)

2. How many trailing zeros does $(2025!)^3$ have?

   Solution: $v_5((2025!)^3) = 3 \cdot v_5(2025!) = 3 \cdot 505 = \mathbf{1515}$. Twos are even more plentiful, so zeros are limited by 5s.

3. What is the smallest integer s so that $5^s \cdot 2025!$ is divisible by 10^{2017}?

   Solution: $v_2(2025!) = 2017$ and $v_5(2025!) = 505$. We need v_5 to reach 2017, so add $s = 2017 - 505 = \mathbf{1512}$ factors of 5. Then $v_2 = v_5 = 2017 \Rightarrow 10^{2017} \mid 5^{1512} \cdot 2025!$.

4. How many trailing zeros does $2^{1000} \cdot 2025!$ have?

   Solution: Multiplying by 2^{1000} increases only the count of 2s, not 5s. The zeros are still limited by $v_5(2025!) = 505$. Hence the product still has **505** trailing zeros.

5. Let k be the number of trailing zeros of 1000!. Find the *last digit* of $7^k + 3^k$.

   Solution: $k = \lfloor\frac{1000}{5}\rfloor + \lfloor\frac{1000}{25}\rfloor + \lfloor\frac{1000}{125}\rfloor + \lfloor\frac{1000}{625}\rfloor = 200 + 40 + 8 + 1 = \mathbf{249}$.
   Modulo 10, both 7 and 3 have cycle length 4. Since $249 \equiv 1 \pmod 4$, we get $7^{249} \equiv 7$ and $3^{249} \equiv 3 \pmod{10}$.
   Sum $\equiv 7 + 3 \equiv \mathbf{0} \pmod{10}$.