Two Classical Irrationality Proofs

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A rational number is a number that can be written as a fraction $\frac{m}{n}$ of integers with $n \neq 0$. We may always assume the fraction is in *lowest terms*, i.e. $\gcd(m, n) = 1$.

$\sqrt{2}$ is irrational

Theorem 1. *$\sqrt{2}$ is irrational.*

Proof. Suppose for contradiction that $\sqrt{2}$ is rational. Then there exist coprime integers m, n with $n \neq 0$ such that

$$\sqrt{2} = \frac{m}{n}. \tag{1}$$

Squaring both sides gives

$$2 = \frac{m^2}{n^2} \quad \Longrightarrow \quad m^2 = 2n^2.$$

Thus m^2 is even, which implies m is even. Write $m = 2k$ for some integer k. Substituting back yields

$$(2k)^2 = 2n^2 \quad \Longrightarrow \quad 4k^2 = 2n^2 \quad \Longrightarrow \quad n^2 = 2k^2.$$

Hence n^2 is even, so n is even. We have shown that both m and n are even, contradicting the assumption that m/n is in lowest terms. Therefore $\sqrt{2}$ cannot be rational. □

Remark. The essence of the argument is that if m^2 is divisible by 2 then so is m, and this forces any putative lowest-terms representation of $\sqrt{2}$ to have a common factor of 2 in numerator and denominator.

$\sqrt[3]{2}$ is irrational

There are several short proofs. We give two standard approaches: a parity-style contradiction and a prime-exponent argument using unique factorization.

Theorem 2. *$\sqrt[3]{2}$ is irrational.*

Proof 1 (lowest-terms contradiction). Assume, for a contradiction, that $\sqrt[3]{2} = \dfrac{m}{n}$ with integers m, n coprime and $n \neq 0$. Cubing yields

$$2 = \frac{m^3}{n^3} \quad \Longrightarrow \quad m^3 = 2n^3.$$

Thus 2 divides m^3; since 2 is prime, it follows that 2 divides m. Write $m = 2k$ for some integer k. Substituting back,

$$(2k)^3 = 2n^3 \quad \Longrightarrow \quad 8k^3 = 2n^3 \quad \Longrightarrow \quad n^3 = 4k^3.$$

Therefore 2 divides n^3, so 2 divides n. Hence m and n are both even, contradicting $\gcd(m, n) = 1$. This contradiction shows $\sqrt[3]{2}$ is irrational. □