Factoring Enrichment

Math 10 · Mr. Merrick · October 22, 2025

1. Factor completely over the real numbers: $x^4 + 1$.
   Complete a "disguised" square and apply a difference of squares:

$$x^4 + 1 = (x^2 + 1)^2 - (\sqrt{2}\,x)^2 = (x^2 - \sqrt{2}\,x + 1)(x^2 + \sqrt{2}\,x + 1).$$

   Each quadratic has discriminant $\Delta = 2 - 4 < 0$, so there are no real linear factors.

2. Factor completely over the real numbers: $x^4 + 4$.
   Note: This is a special case of the Sophie Germain identity, named for Sophie Germain.
   Add and subtract $4x^2$ to form a difference of squares:

$$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2).$$

   (General form: $a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$.)

3. Factor completely over the real numbers: $x^8 - 1$.
   Apply successive differences of squares and use the factorization of $x^4 + 1$:

$$x^8 - 1 = (x^4 - 1)(x^4 + 1) = (x^2 - 1)(x^2 + 1)(x^4 + 1)$$
$$= (x - 1)(x + 1)(x^2 + 1)(x^2 - \sqrt{2}\,x + 1)(x^2 + \sqrt{2}\,x + 1).$$

4. Factor completely over the real numbers: $x^6 - 64$.
Let $u = x^2$. Then $x^6 - 64 = u^3 - 4^3 = (u-4)(u^2+4u+16)$. Substitute back and factor each factor over $\mathbb{R}$:
$$x^6 - 64 = (x^2-4)(x^4+4x^2+16) = (x-2)(x+2)\big((x^2+4)^2-(2x)^2\big)$$
$$= (x-2)(x+2)(x^2-2x+4)(x^2+2x+4).$$

5. Factor completely over the real numbers: $x^{12} - y^{12}$.
Use difference of squares, cubes, and then real quadratic factors:
$$x^{12}-y^{12} = (x^6-y^6)(x^6+y^6) = (x^3-y^3)(x^3+y^3)(x^2+y^2)(x^4-x^2y^2+y^4)$$
$$= (x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)(x^2+y^2)(x^4-x^2y^2+y^4).$$
Over $\mathbb{R}$, the last quartic splits further:
$$x^4-x^2y^2+y^4 = (x^2+\sqrt{3}\,xy+y^2)(x^2-\sqrt{3}\,xy+y^2).$$
Final result:
$$(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)(x^2+y^2)(x^2+\sqrt{3}\,xy+y^2)(x^2-\sqrt{3}\,xy+y^2).$$

6. Determine whether each polynomial factors over the real numbers. If so, factor completely.
(a) x^4-5x^2+4 (b) x^4+4x^2+4 (c) x^4+4x^2+5
(a) Let $u = x^2$: $u^2-5u+4 = (u-1)(u-4)$. Hence $(x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$.
(b) Perfect square: $x^4+4x^2+4 = (x^2+2)^2$.
(c) Suppose $(x^2+ax+b)(x^2-ax+b) = x^4+(2b-a^2)x^2+b^2$. Match coefficients: $2b-a^2=4$, $b^2=5$. Taking $b=\sqrt{5}$ gives $a^2 = 2\sqrt{5}-4 > 0$, so
$$x^4+4x^2+5 = \big(x^2+\sqrt{2\sqrt{5}-4}\,x+\sqrt{5}\big)\big(x^2-\sqrt{2\sqrt{5}-4}\,x+\sqrt{5}\big).$$
Over $\mathbb{Q}$ this is irreducible, but over $\mathbb{R}$ it splits into these quadratics.