Relations and Functions
Booklet 2: Functions

Mr. Merrick · December 8, 2025

Contents

Functions: Intuition and Formal Definition

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Explainer

Goal. Bridge intuitive and formal definitions of functions using relations.

Function intuition. A function takes an input and produces *exactly one* output.
Formal definition. A function $f : A \to B$ is a relation $f \subseteq A \times B$ such that:

- Every $a \in A$ is the first coordinate of some pair in f (at least one output).
- Each $a \in A$ appears at most once as a first coordinate (at most one output).

Vocabulary.

- **Domain** $= A$ (inputs).
- **Codomain** $= B$ (allowed outputs).
- **Range** $=$ actual outputs:

$$\operatorname{ran}(f) = \{b \in B : \exists a, (a, b) \in f\}.$$

1. Function or not? (Ordered pairs)

(a) Is the following a function $A \to B$?

$$f = \{(1, x), (2, y), (3, y)\}, \quad A = \{1, 2, 3\}.$$

(b) Is this a function?

$$g = \{(1, a), (1, b), (2, a)\}.$$

2. Domain, codomain, and range

(c) Let $h : \mathbb{Z} \to \mathbb{Z}$ be $h(n) = n^2$. State domain, codomain, and range.

(d) Let $k : \mathbb{Z} \to \mathbb{W}$ be defined by $k(n) = n^2$. State the domain, codomain, and range.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \sqrt{x}$. Is this a function? If not, fix it.

3. Representing functions (tables, mappings, graphs)

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Goal. See the *same* function in several different representations.

1. Tables. List input–output pairs:

x	$f(x)$
0	1
1	3
2	5

2. Mapping diagrams. Draw dots for inputs and outputs, with arrows showing how each input maps to an output.

3. Graphs. Plot the points $(x, f(x))$ on a coordinate plane. A graph represents a function of x if it passes the **vertical line test**: no vertical line intersects the graph more than once.

Examples

(f) Consider the table

x	$f(x)$
-1	2
0	1
2	5

Is this a function?

(g) Consider the table

x	$g(x)$
0	4
1	5
0	6

Is this a function?

4. Simple proofs about functions

(h) Prove: If $(a, b), (a, c) \in f$ and f is a function, then $b = c$.

(i) Explain why there is exactly one function $\emptyset \to B$ for any set B.

(j) Give two different functions $f, g : \{1, 2\} \to \{0, 1\}$ with the same range.

5. Extra practice: functions and representations

(k) Use a mapping diagram to show a function $f : \{1, 2, 3\} \to \{a, b\}$ where

$$f(1) = a, \quad f(2) = a, \quad f(3) = b.$$

Then write f as a set of ordered pairs and as a table.

(l) Sketch a graph of $y = x^2$ and use the vertical line test to explain why it represents a function of x.

(m) Sketch a graph of $x = y^2$ and explain why it is *not* a function of x.

Injective, Surjective, and Bijective Functions

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Goal. Classify functions $f : A \to B$ by how they use their codomain B.

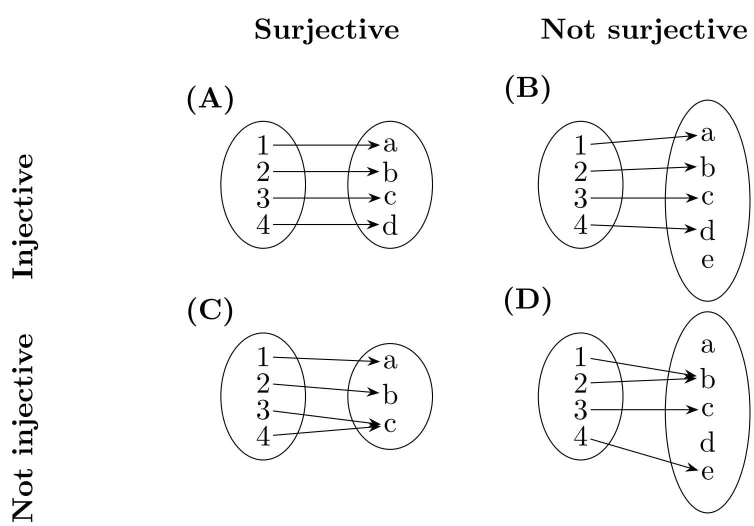
Definitions.

- **Injective** (one-to-one):
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$
No two different inputs share an output.
- **Surjective** (onto):
$$\mathrm{ran}(f) = B.$$
Every element of the codomain is hit at least once.
- **Bijective**: Both injective and surjective.

Mapping-diagram view.

- Injective: arrows never land on the same output dot.
- Surjective: every output dot has at least one arrow landing on it.
- Bijective: every input arrow lands on a distinct output, and every output is used.

1. Mapping-diagram examples



Summary: (A) injective & surjective (bijective) (B) injective, not surjective (C) surjective, not injective (D) neither injective nor surjective.

2. Classifying small functions

(a) Let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be $f(1) = a$, $f(2) = b$, $f(3) = c$. Is f injective? Surjective?

(b) Let $g : \{1, 2, 3, 4\} \to \{x, y\}$ be $g(1) = x$, $g(2) = x$, $g(3) = y$, $g(4) = y$.

3. Linear functions on $\mathbb{R}$

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be $h(x) = 3x - 5$. Prove injective and surjective.

(d) Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(n) = 2n$.

4. (Enrichment) Deeper properties

(e) Let $f : A \to B$ be bijective. Explain in words why the *inverse* relation

$$f^{-1} = \{(b, a) : (a, b) \in f\}$$

is actually a function from B to A.

(f) Let A, B be finite with $|A| = |B|$. Prove: If $f : A \to B$ is injective, then f is surjective.

(g) **Big idea: comparing the sizes of infinite sets.** Mathematicians say two sets A and B have the same *size* (the same **cardinality**) if there is a bijection $f : A \to B$. Explain how this idea applies to the integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$.

5. Extra practice: classify functions via diagrams

(h) A mapping diagram shows $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ with arrows $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$. Is the function injective? Surjective? Bijective?

(i) Another diagram shows $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ with $1 \mapsto a$, $2 \mapsto a$, $3 \mapsto b$, $4 \mapsto c$. Classify the function.

(j) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 1$. Is f injective? Surjective?

Even and Odd Functions

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Goal. Understand two important types of symmetry in functions: even and odd functions.

Definitions.

- A function f is **even** if
$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain.}$$
Graphically, even functions are symmetric across the y-axis.
- A function f is **odd** if
$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain.}$$
Graphically, odd functions have symmetry about the origin (rotational symmetry 180°).

Key observations.

- A function can be even, odd, both (only the zero function), or neither.
- Checking even/odd-ness is done by substituting $-x$ and simplifying.

1. Examples of even and odd functions

(a) Show that $f(x) = x^2$ is even.

(b) Show that $g(x) = x^3$ is odd.

(c) Determine whether $h(x) = x^2 + 3$ is even, odd, or neither.

(d) Determine whether $p(x) = x^3 + 2x$ is even, odd, or neither.

2. Proving whether a function is even or odd

(e) Determine whether
$$f(x) = \frac{1}{x^2}$$
is even, odd, or neither.

(f) Determine whether

$$g(x) = \frac{x}{x^2 + 1}$$

is even, odd, or neither.

(g) For $h(x) = x^2 + x$, show that it is neither even nor odd.

3. Summary and strategy

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How to check if a function is even or odd

1. Substitute $-x$ into the function.
2. Simplify completely.
3. Compare the result with $f(x)$ and $-f(x)$:

If $f(-x) = f(x)$, then the function is **even**.

If $f(-x) = -f(x)$, then the function is **odd**.

Otherwise, it is **neither**.

Graphical intuition.

- Even functions: symmetric across the y-axis.
- Odd functions: symmetric about the origin (rotate the graph 180°).

Relations vs Functions: Mixed Representations

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Goal. Bring everything together: determine whether a given representation (ordered pairs, mapping diagram, table, or graph) defines a function.

Key test. Each input must have **exactly one** output.
Representations.

- **Ordered pairs:** list of (x, y) values.
- **Mapping diagram:** arrows from inputs to outputs.
- **Table:** two-column list of x and $f(x)$.
- **Graph:** points $(x, f(x))$ in the plane. Use the **vertical line test**: if some vertical line hits the graph twice, the relation is not a function of x.

1. Ordered pairs

(a) $R = \{(1, 2), (2, 3), (3, 4)\}$. Function?

(b) $S = \{(1, a), (1, b), (2, a)\}$. Function?

2. Mapping diagrams

(c) $1 \mapsto x$, $2 \mapsto x$, $3 \mapsto y$.

(d) $a \mapsto 1$, $a \mapsto 2$.

3. Tables

(e)

x	$f(x)$
1	1
2	1
3	1

(f)

x	$g(x)$
1	2
1	3

4. Graphs

(g) Vertical line $x = 3$. Function?

(h) Graph of $y = x^2$. Function?

(i) Graph of a sideways parabola $x = y^2$.

5. Composition & classification

(j) Let $f = \{(1, 2), (2, 3), (3, 4)\}$ and $g = \{(2, a), (3, b), (4, c)\}$. Compute $g \circ f = g(f(x))$.

(k) Is $g \circ f$ injective? Surjective (onto $\{a, b, c\}$)?

6. Extra practice: spotting functions

(l) Decide whether each relation is a function from $\mathbb{R}$ to $\mathbb{R}$:

i. $y = 3x + 1$

ii. $x^2 + y^2 = 1$

iii. $y^2 = x$

(m) A table shows

x	$h(x)$
-2	4
-1	1
0	0
1	1
2	4

Is h a function? Explain using both the table and the idea of a graph.