Relations and Functions
Booklet 2: Functions

Mr. Merrick · December 8, 2025

Contents

Functions: Intuition and Formal Definition

Mr. Merrick · December 8, 2025

Explainer

Goal. Bridge intuitive and formal definitions of functions using relations.

Function intuition. A function takes an input and produces *exactly one* output.
Formal definition. A function $f : A \to B$ is a relation $f \subseteq A \times B$ such that:

- Every $a \in A$ is the first coordinate of some pair in f (at least one output).
- Each $a \in A$ appears at most once as a first coordinate (at most one output).

Vocabulary.

- **Domain** $= A$ (inputs).
- **Codomain** $= B$ (allowed outputs).
- **Range** = actual outputs:

$$\operatorname{ran}(f) = \{b \in B : \exists a, (a, b) \in f\}.$$

1. Function or not? (Ordered pairs)

(a) Is the following a function $A \to B$?

$$f = \{(1, x), (2, y), (3, y)\}, \quad A = \{1, 2, 3\}.$$

Solution. Yes — each input has exactly one output.

(b) Is this a function?

$$g = \{(1, a), (1, b), (2, a)\}.$$

Solution. No — input 1 has two outputs.

2. Domain, codomain, and range

(c) Let $h : \mathbb{Z} \to \mathbb{Z}$ be $h(n) = n^2$. State domain, codomain, and range.

Solution. Domain $= \mathbb{Z}$. Codomain $= \mathbb{Z}$. Range $= \{0, 1, 4, 9, 16, \dots\}$.

(d) Let $k : \mathbb{Z} \to \mathbb{W}$ be defined by $k(n) = n^2$. State the domain, codomain, and range.

Solution. Domain $= \mathbb{Z}$ (all integers). Codomain $= \mathbb{W}$ (whole numbers). Range $= \{n^2 : n \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, \dots\} \subseteq \mathbb{W}$.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \sqrt{x}$. Is this a function? If not, fix it.

Solution. Not a function on all $\mathbb{R}$ because negatives have no real square root. As $f : [0, \infty) \to \mathbb{R}$ it *is* a function.

3. Representing functions (tables, mappings, graphs)

Explainer

Goal. See the *same* function in several different representations.

1. Tables. List input–output pairs:

x	$f(x)$
0	1
1	3
2	5

2. Mapping diagrams. Draw dots for inputs and outputs, with arrows showing how each input maps to an output.
3. Graphs. Plot the points $(x, f(x))$ on a coordinate plane. A graph represents a function of x if it passes the **vertical line test**: no vertical line intersects the graph more than once.

Examples

(f) Consider the table

x	$f(x)$
-1	2
0	1
2	5

Is this a function?

Solution. Yes. Each input x appears at most once, so each has exactly one output.

(g) Consider the table

x	$g(x)$
0	4
1	5
0	6

Is this a function?

Solution. No. Input 0 has two different outputs, 4 and 6.

4. Simple proofs about functions

(h) Prove: If $(a, b), (a, c) \in f$ and f is a function, then $b = c$.

Solution. Otherwise a would have two distinct outputs, contradicting the definition.

(i) Explain why there is exactly one function $\emptyset \to B$ for any set B.

Solution. The domain is empty; there are no inputs to assign. The empty set of ordered pairs satisfies the function rule.

(j) Give two different functions $f, g : \{1, 2\} \to \{0, 1\}$ with the same range.

Solution. Example: $f(1) = 0, f(2) = 1$ and $g(1) = 1, g(2) = 0$. Both have range $\{0, 1\}$.

5. Extra practice: functions and representations

(k) Use a mapping diagram to show a function $f : \{1, 2, 3\} \to \{a, b\}$ where

$$f(1) = a, \quad f(2) = a, \quad f(3) = b.$$

Then write f as a set of ordered pairs and as a table.

Solution. Mapping diagram: three dots $\{1, 2, 3\}$ on the left, two dots $\{a, b\}$ on the right, with arrows $1 \to a$, $2 \to a$, $3 \to b$. Ordered pairs: $\{(1, a), (2, a), (3, b)\}$. Table:

x	$f(x)$
1	a
2	a
3	b

(l) Sketch a graph of $y = x^2$ and use the vertical line test to explain why it represents a function of x.

Solution. The graph is the standard parabola opening upward. Any vertical line $x = c$ intersects it in at most one point; hence each input x has at most one y-value.

(m) Sketch a graph of $x = y^2$ and explain why it is *not* a function of x.

Solution. The graph is a sideways parabola. For $x > 0$, the vertical line $x = c$ intersects the graph twice (once with positive y, once with negative y), so some inputs x would have two outputs y.

Injective, Surjective, and Bijective Functions

Mr. Merrick · December 8, 2025

Explainer

Goal. Classify functions $f : A \to B$ by how they use their codomain B.

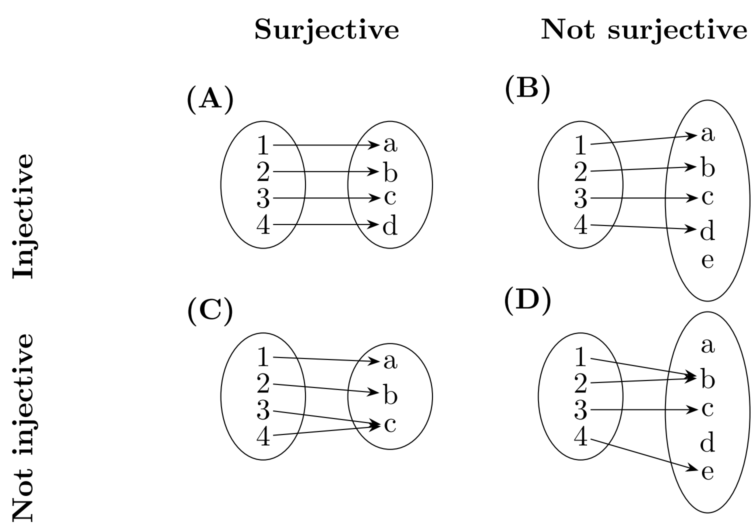
Definitions.

- **Injective** (one-to-one):
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$
No two different inputs share an output.
- **Surjective** (onto):
$$\mathrm{ran}(f) = B.$$
Every element of the codomain is hit at least once.
- **Bijective**: Both injective and surjective.

Mapping-diagram view.

- Injective: arrows never land on the same output dot.
- Surjective: every output dot has at least one arrow landing on it.
- Bijective: every input arrow lands on a distinct output, and every output is used.

1. Mapping-diagram examples



Summary: (A) injective & surjective (bijective) (B) injective, not surjective (C) surjective, not injective (D) neither injective nor surjective.

2. Classifying small functions

(a) Let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be $f(1) = a$, $f(2) = b$, $f(3) = c$. Is f injective? Surjective?

Solution. Injective: Yes — outputs distinct. Surjective: No — d unused.

(b) Let $g : \{1, 2, 3, 4\} \to \{x, y\}$ be $g(1) = x$, $g(2) = x$, $g(3) = y$, $g(4) = y$.

Solution. Injective: No (1 and 2 share output; 3 and 4 share output). Surjective: Yes — both x and y appear.

3. Linear functions on $\mathbb{R}$

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be $h(x) = 3x - 5$. Prove injective and surjective.

Solution. Injective: if $3x_1 - 5 = 3x_2 - 5 \Rightarrow x_1 = x_2$. Surjective: For any $y \in \mathbb{R}$, the equation $3x - 5 = y$ has solution $x = (y + 5)/3 \in \mathbb{R}$. Thus h is bijective.

(d) Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(n) = 2n$.

Solution. Injective: Yes — if $2n_1 = 2n_2$ then $n_1 = n_2$. Surjective: No — odd integers are missing. If the codomain is changed to $2\mathbb{Z}$, it becomes surjective and thus bijective.

4. (Enrichment) Deeper properties

(e) Let $f : A \to B$ be bijective. Explain in words why the *inverse* relation

$$f^{-1} = \{(b, a) : (a, b) \in f\}$$

is actually a function from B to A.

Solution. Because f is surjective, every element $b \in B$ is the output of f for *at least one* input $a \in A$ (there exists some a with $f(a) = b$). Because f is injective, no two different inputs in A can share the same output in B, so there is *at most one* such a. Put together, each $b \in B$ is paired with *exactly one* $a \in A$. Flipping the pairs therefore gives a function $f^{-1} : B \to A$.

(f) Let A, B be finite with $|A| = |B|$. Prove: If $f : A \to B$ is injective, then f is surjective.

Solution. Injectivity means different elements of A always go to different elements of B. So the range of f has exactly $|A|$ elements. But $|A| = |B|$, so the range has $|B|$ elements and therefore *must* be all of B. Thus f is surjective.

(g) **Big idea: comparing the sizes of infinite sets.** Mathematicians say two sets A and B have the same *size* (the same **cardinality**) if there is a bijection $f : A \to B$. Explain how this idea applies to the integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$.

Solution. Even though there are "many more" rational numbers than integers if you look on the number line, it turns out there is a bijection between $\mathbb{Z}$ and $\mathbb{Q}$. One way to see this is:

- Write all rational numbers as fractions m/n with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ in an infinite grid (rows indexed by m, columns by n).
- Walk through this grid in a zigzag (diagonal) pattern so that every grid position is eventually visited.
- Whenever you land on a fraction, simplify it and *skip* it if you have already seen that simplified value.
- Label the first new rational you meet with 0, the next new one with 1, the next with -1, then 2, -2, and so on, using all of $\mathbb{Z}$.

In this way each integer is paired with exactly one rational number, and every rational number appears somewhere on the list. That pairing is a bijection $\mathbb{Z} \to \mathbb{Q}$, so by our definition $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality.

5. Extra practice: classify functions via diagrams

(h) A mapping diagram shows $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ with arrows $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$. Is the function injective? Surjective? Bijective?

Solution. Each output is hit exactly once, so the function is injective and surjective; therefore bijective.

(i) Another diagram shows $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ with $1 \mapsto a$, $2 \mapsto a$, $3 \mapsto b$, $4 \mapsto c$. Classify the function.

Solution. Two different inputs share the output a, so it is not injective. Every element of B is used, so it is surjective.

(j) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 1$. Is f injective? Surjective?

Solution. Injective: Yes, because $n_1 + 1 = n_2 + 1$ forces $n_1 = n_2$.

Surjective (onto $\mathbb{N}$): No. The number 1 is in the codomain, but there is no $n \in \mathbb{N}$ such that $n + 1 = 1$. So 1 never appears as an output of the function.

If the codomain were $\{2, 3, 4, \dots\}$ instead, then every element of the codomain would appear as an output, and the function would be bijective.

Even and Odd Functions

Mr. Merrick · December 8, 2025

Explainer

Goal. Understand two important types of symmetry in functions: even and odd functions.

Definitions.

- A function f is **even** if

$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain.}$$

  Graphically, even functions are symmetric across the y-axis.

- A function f is **odd** if

$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain.}$$

  Graphically, odd functions have symmetry about the origin (rotational symmetry 180°).

Key observations.

- A function can be even, odd, both (only the zero function), or neither.
- Checking even/odd-ness is done by substituting $-x$ and simplifying.

1. Examples of even and odd functions

(a) Show that $f(x) = x^2$ is even.

Solution. Compute $f(-x) = (-x)^2 = x^2 = f(x)$. Since $f(-x) = f(x)$ for all x, the function is even.

(b) Show that $g(x) = x^3$ is odd.

Solution. Compute $g(-x) = (-x)^3 = -x^3 = -g(x)$. Since $g(-x) = -g(x)$ for all x, the function is odd.

(c) Determine whether $h(x) = x^2 + 3$ is even, odd, or neither.

Solution. $h(-x) = (-x)^2 + 3 = x^2 + 3 = h(x)$, so h is even.

(d) Determine whether $p(x) = x^3 + 2x$ is even, odd, or neither.

Solution. $p(-x) = (-x)^3 + 2(-x) = -x^3 - 2x = -(x^3 + 2x) = -p(x)$, so p is odd.

2. Proving whether a function is even or odd

(e) Determine whether

$$f(x) = \frac{1}{x^2}$$

is even, odd, or neither.

Solution. Compute

$$f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x).$$

So f is even.

(f) Determine whether

$$g(x) = \frac{x}{x^2+1}$$

is even, odd, or neither.

Solution. Compute

$$g(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -\frac{x}{x^2+1} = -g(x).$$

So g is odd.

(g) For $h(x) = x^2 + x$, show that it is neither even nor odd.

Solution. Compute:

$$h(-x) = (-x)^2 + (-x) = x^2 - x.$$

Compare:

$$h(x) = x^2 + x, \qquad h(-x) = x^2 - x.$$

Neither $h(-x) = h(x)$ nor $h(-x) = -h(x)$, so the function is neither even nor odd.

3. Summary and strategy

Explainer

How to check if a function is even or odd

1. Substitute $-x$ into the function.
2. Simplify completely.
3. Compare the result with $f(x)$ and $-f(x)$:

If $f(-x) = f(x)$, then the function is **even**.

If $f(-x) = -f(x)$, then the function is **odd**.

Otherwise, it is **neither**.

Graphical intuition.

- Even functions: symmetric across the y-axis.
- Odd functions: symmetric about the origin (rotate the graph 180°).

Relations vs Functions: Mixed Representations

Mr. Merrick · December 8, 2025

Explainer

Goal. Bring everything together: determine whether a given representation (ordered pairs, mapping diagram, table, or graph) defines a function.

Key test. Each input must have **exactly one** output.
Representations.

- **Ordered pairs:** list of (x, y) values.
- **Mapping diagram:** arrows from inputs to outputs.
- **Table:** two-column list of x and $f(x)$.
- **Graph:** points $(x, f(x))$ in the plane. Use the **vertical line test**: if some vertical line hits the graph twice, the relation is not a function of x.

1. Ordered pairs

(a) $R = \{(1, 2), (2, 3), (3, 4)\}$. Function?

Solution. Yes — each input used once.

(b) $S = \{(1, a), (1, b), (2, a)\}$. Function?

Solution. No — 1 has two outputs.

2. Mapping diagrams

(c) $1 \mapsto x$, $2 \mapsto x$, $3 \mapsto y$.

Solution. Function (many-to-one is allowed).

(d) $a \mapsto 1$, $a \mapsto 2$.

Solution. Not a function — a has two outputs.

3. Tables

(e)

x	$f(x)$
1	1
2	1
3	1

Solution. Function — each input appears once.

(f)

x	$g(x)$
1	2
1	3

Solution. Not a function — repeated input with conflicting outputs.

4. Graphs

(g) Vertical line $x = 3$. Function?

Solution. No — for $x = 3$ there are infinitely many y-values; fails vertical line test.

(h) Graph of $y = x^2$. Function?

Solution. Yes — each x has one y; passes vertical line test.

(i) Graph of a sideways parabola $x = y^2$.

Solution. Not a function of x — some x have two y values; fails vertical line test.

5. Composition & classification

(j) Let $f = \{(1,2),(2,3),(3,4)\}$ and $g = \{(2,a),(3,b),(4,c)\}$. Compute $g \circ f = g(f(x))$.

Solution. $1 \mapsto 2 \mapsto a$, $2 \mapsto 3 \mapsto b$, $3 \mapsto 4 \mapsto c$. So $g \circ f = \{(1,a),(2,b),(3,c)\}$.

(k) Is $g \circ f$ injective? Surjective (onto $\{a,b,c\}$)?

Solution. Injective: Yes — outputs distinct. Surjective: Yes — all $\{a,b,c\}$ appear.

6. Extra practice: spotting functions

(l) Decide whether each relation is a function from $\mathbb{R}$ to $\mathbb{R}$:

i. $y = 3x + 1$

ii. $x^2 + y^2 = 1$

iii. $y^2 = x$

Solution. (i) Function — each x gives exactly one y. (ii) Not a function — the circle $x^2 + y^2 = 1$ fails the vertical line test. (iii) Not a function of x (sideways parabola).

(m) A table shows

x	$h(x)$
-2	4
-1	1
0	0
1	1
2	4

Is h a function? Explain using both the table and the idea of a graph.

Solution. Yes — each input appears once with a single output. If we plot the points, they lie on the graph of $y = x^2$, which we know is a function.