Relations and Functions

Booklet 1: Sets, Subsets, Cartesian Products, and Relations

Mr. Merrick · December 8, 2025

Contents

Intro to Sets, Subsets, and Basic Notation

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Explainer

Goal. Build a foundation for all later work on relations and functions by learning the language of sets.

Key ideas.

- A **set** is a collection of distinct objects (elements).
- In **roster notation**, list elements explicitly:

$$A = \{1, 2, 3\}.$$

- Sets *ignore duplicates* and *ignore order.*
- $x \in A$ means x is an element of A.
- A **subset** $A \subseteq B$ means every element of A is in B.
- A **proper subset** $A \subset B$ means $A \subseteq B$ but $A \neq B$.

Worked example. Is $\{1, 2, 2, 3\} = \{3, 2, 1\}$? Yes — order and duplicates do not matter.

1. Identifying sets

(a) Which of the following represent valid sets? (i) $\{1, 2, 3\}$ (ii) $\{a, b, c, b\}$ (iii) $(1, 2, 3)$ (iv) $\{\text{cat}, \text{dog}, 7\}$

(b) List all elements of

$$A = \{3, 1, 4, 1, 5, 9, 2, 6\}.$$

2. Membership and subsets

(c) Let $A = \{1, 3, 5, 7\}$. Determine whether:

i. $5 \in A$

ii. $2 \in A$

iii. $\{3, 7\} \subseteq A$

iv. $\{1, 2\} \subseteq A$

(d) Compare $B = \{x, y, z\}$ and $C = \{z, x\}$. Determine which are true:

i. $C \subseteq B$

ii. $B \subseteq C$

iii. $C \subset B$

iv. $B = C$

3. Unusual elements of sets

(e) Let

$$X = \{\emptyset, 1, \{1\}\}.$$

Identify each element and answer: Is $\emptyset \in X$? Is $\emptyset \subseteq X$?

(f) Compare

$$A = \{1, 2, \{3\}\}, \qquad B = \{1, 2, 3\}.$$

Are they equal?

4. Quick constructions

(g) Create a set with 5 elements: an integer, a decimal, a letter, a word, and a set.

(h) Let

$$A = \{n \in \mathbb{Z} : -2 \leq n \leq 3\}.$$

Convert to roster notation.

5. Subset counting

(i) How many subsets does $A = \{1, 2, 3, 4\}$ have?

6. Extra practice: sets and subsets

(j) Decide whether the following pairs of sets are equal. If not, explain why. (i) $\{1, 2, 3\}$ and $\{3, 2, 1, 1\}$ (ii) $\{a, \{b\}\}$ and $\{a, b\}$ (iii) $\{\emptyset\}$ and $\emptyset$

(k) Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$. List three different subsets of U that contain A as a subset.

(l) Write each in roster notation: (i) $\{n \in \mathbb{Z} : 0 \leq n \leq 5\}$ (ii) $\{n \in \mathbb{Z} : -3 < n < 2\}$

(m) Describe in set-builder notation the set

$$B = \{-5, -3, -1, 1, 3, 5\}.$$

Set-Builder Notation, Empty Set, and Power Sets

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Goal. Move between roster and set-builder notation and understand the empty set more formally.

Key ideas.

- Set-builder notation describes elements using a condition:

$$A = \{x \in \mathbb{Z} : x \text{ is odd}\}.$$

- The symbol ":" reads "such that."
- The **empty set** $\emptyset$ has no elements.
- The **power set** $\mathcal{P}(A)$ is the set of all subsets of A.

Worked example. $\{2, 4, 6, 8\} = \{n \in \mathbb{Z} : 2 \leq n \leq 8,\ n \text{ even}\}$.

1. Roster ↔ builder

(a) Convert to builder notation:

$$A = \{1, 3, 5, 7, 9\}.$$

(b) Convert to roster form:

$$B = \{n \in \mathbb{Z} : -3 \leq n \leq 2\}.$$

(c) Fix the domain:

$$C = \{x : x > 0\}.$$

2. Empty set basics

(d) Which describe $\emptyset$?

i. $\{x \in \mathbb{R} : x^2 = -1\}$

ii. $\{n \in \mathbb{Z} : n \text{ even and odd}\}$

iii. $\{x \in \mathbb{R} : x < 0 \text{ and } x > 10\}$

(e) Is $\emptyset \subseteq \{\emptyset\}$? Is $\emptyset \in \{\emptyset\}$?

3. Power sets

(f) List all subsets of $A = \{x, y, z\}$.

(g) How many subsets does a set of size n have?

4. Why 2^n?

(h) Explain the reasoning behind $|\mathcal{P}(A)| = 2^{|A|}$.

5. Extra practice: builder notation and power sets

(i) Write the set in roster form:

$$D = \{n \in \mathbb{Z} : -2 \leq n < 3\}.$$

(j) Write in set-builder notation:

$$E = \{-4, -1, 2, 5, 8\}.$$

(k) Let $B = \{1, 2\}$. Write $\mathcal{P}(B)$ and then $|\mathcal{P}(B)|$.

(l) Give a set-builder description of the empty set that looks different from parts (i)–(iii) above.

Set-Builder Notation vs. Roster Notation

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Goal. Learn how to write sets in two different ways:

- **Roster notation:** list all elements explicitly,

$$A = \{1, 3, 5, 7\}.$$

- **Set-builder notation:** describe elements using a rule,

$$A = \{n \in \mathbb{Z} : n \text{ is an odd integer from 1 to 7}\}.$$

Key ideas.

- Roster is good for small, finite sets.
- Set-builder is good when:
  - the set is infinite,
  - the set follows a pattern,
  - the property is easier to describe than enumerate.
- The colon ":" can be read as "such that."
- Set-builder notation uses a membership requirement:

$$\{x \in (\text{universe}) : \text{condition on } x\}.$$

Worked example. Convert the set $A = \{2, 4, 6, 8\}$ into set-builder notation:

$$A = \{n \in \mathbb{Z} : n \text{ is even and } 2 \leq n \leq 8\}.$$

1. Converting roster → set-builder

(a) Convert the following set to set-builder notation:

$$A = \{1, 3, 5, 7, 9\}.$$

(b) Write the set-builder form of:

$$B = \{-4, -2, 0, 2, 4\}.$$

(c) Convert to set-builder notation:

$$C = \{\text{red}, \text{blue}, \text{green}\}.$$

2. Converting set-builder → roster

(d) Convert the set

$$D = \{n \in \mathbb{Z} : 0 < n < 6\}$$

into roster notation.

(e) Convert

$$E = \{x \in \mathbb{R} : x^2 = 4\}.$$

(f) Convert the following set:

$$F = \{n \in \mathbb{Z} : n \text{ is a multiple of 3 and } -10 \leq n \leq 10\}.$$

3. Identifying mistakes in set-builder notation

(g) Consider the notation

$$G = \{x : x > 0\}.$$

What is missing?

(h) Identify the error in:

$$H = \{n \in \mathbb{Z} : n = \text{even}\}.$$

(i) Determine whether the following set is well-defined:

$$J = \{x \in \mathbb{R} : x \text{ is a big number}\}.$$

4. Mixed practice

(j) Write the set of all integers that are multiples of 4 (use builder notation).

(k) Give the roster notation for the set:

$$K = \{n \in \mathbb{Z} : n \text{ is prime and } 1 < n < 20\}.$$

(l) Convert the set into roster form:

$$L = \{x \in \mathbb{R} : x^2 - 9 = 0\}.$$

(m) Convert to set-builder notation:

$$M = \{-5, -3, -1, 1, 3, 5\}.$$

5. Extra practice: more conversions

(n) Convert to roster form:

$$S = \{n \in \mathbb{Z} : -1 \leq n \leq 4,\ n \text{ even}\}.$$

(o) Convert to set-builder notation:

$$T = \{10, 20, 30, 40, \dots\}.$$

(p) Write a set-builder description of all real numbers between -3 and 5, including endpoints.

The Empty Set and Vacuous Truth

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Goal. Deepen our understanding of $\emptyset$ and why statements about "all elements of $\emptyset$" are automatically true.

Key idea: Vacuous truth. A universal statement about the elements of $\emptyset$ is true because there are no counterexamples.
Example. "All unicorns are blue" is (logically) true — there are no unicorns to violate it.

1. Basic truth statements

(a) Determine whether each is true:

i. $\emptyset \in \emptyset$

ii. $\emptyset \subseteq \emptyset$

iii. $\{\emptyset\} \subseteq \emptyset$

(b) Is the statement "Every element of $\emptyset$ is prime" true?

2. Builder-notation emptiness

(c) Which sets are empty?

i. $\{x \in \mathbb{R} : x^2 = 4\}$

ii. $\{x \in \mathbb{R} : x^2 = 5\}$

iii. $\{x \in \mathbb{R} : x^2 = -9\}$

3. Universal statements over empty sets

(d) Explain why

$$(\forall x \in \emptyset)\ x > 1000$$

is true.

(e) Provide an example of a false *existential* statement involving $\emptyset$.

4. Extra practice: truth with $\emptyset$

(f) Decide whether each statement is true or false. Briefly justify. (i) $\forall x \in \emptyset,\ x^2 > 0$ (ii) $\exists x \in \emptyset,\ x^2 = 1$ (iii) $\exists x \in \emptyset,\ x$ is an integer

(g) Give a real-world "vacuously true" statement (like the unicorn example) and explain why it is vacuously true.

Cartesian Products and Grids

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Goal. Understand ordered pairs and the structure of $A \times B$ as the foundation for defining relations and functions.

Key ideas.

- $A \times B = \{(a, b) : a \in A,\ b \in B\}$.
- Order matters: $(a, b) \neq (b, a)$ in general.
- $|A \times B| = |A| \cdot |B|$.
- If A or B is empty, $A \times B = \emptyset$.

1. Listing and counting

(a) Let $A = \{1, 2\}$ and $B = \{x, y, z\}$.

i. List $A \times B$.

ii. How many elements does $A \times B$ have?

(b) Without listing, compute $|P \times Q|$ if $|P| = 5$ and $|Q| = 7$.

2. Empty-set cases

(c) Compute $A \times \emptyset$.

(d) Compute $\emptyset \times B$.

3. Grid interpretation

(e) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Draw or describe $A \times B$ as a table or grid.

4. Product proofs

(f) Prove: If $A \subseteq B$ then $A \times C \subseteq B \times C$.

(g) Provide a counterexample showing the converse need not hold.

(h) Prove:

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

5. Extra practice: more products

(i) Let $A = \{0, 1, 2\}$ and $B = \{p, q\}$. List all elements of $A \times B$ and of $B \times A$. Are they the same set?

(j) If $|A| = 3$, $|B| = 4$, and $|C| = 2$, compute $|A \times B \times C|$.

(k) Suppose A has 5 elements. How many elements does $A \times A$ have? What about $A \times A \times A$?

Relations as Subsets of Cartesian Products

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Goal. Understand relations as sets of ordered pairs taken from a Cartesian product. This prepares us to define functions as *special* relations.

Key ideas.

- A relation R from A to B is any subset of $A \times B$:

$$R \subseteq A \times B.$$

- The **domain** of R:

$$\mathrm{dom}(R) = \{a \in A : \exists b,\ (a, b) \in R\}.$$

- The **range** of R:

$$\mathrm{ran}(R) = \{b \in B : \exists a,\ (a, b) \in R\}.$$

- The **inverse relation**:

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Worked example. If $R = \{(1, a), (3, b)\}$ from $\{1, 2, 3\}$ to $\{a, b\}$ then $\mathrm{dom}(R) = \{1, 3\}$ and $\mathrm{ran}(R) = \{a, b\}$.

1. Identifying relations

(a) Let $A = \{1, 2\}$, $B = \{x, y, z\}$ and

$$R = \{(1, y), (2, x)\}.$$

Is $R \subseteq A \times B$?

(b) Let

$$S = \{(x, 1), (2, 1)\}, \qquad A = \{x, y\}, \quad B = \{1, 2, 3\}.$$

Is S a relation from A to B?

2. Domain and range

(c) Let

$$R = \{(1, a), (1, b), (2, a), (3, c)\}.$$

Find $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$.

(d) Let

$$T = \{(m, n) \in \mathbb{Z}^2 : m < n\}.$$

Describe the domain and range.

3. Inverse relations

(e) Let

$$R = \{(1, a), (2, b), (3, b)\}.$$

Compute R^{-1}.

(f) Let

$$D = \{(m, n) \in \mathbb{Z}^2 : m \mid n\}.$$

Describe D^{-1} in words.

4. Relation proofs

(g) Prove that R^{-1} is a relation from B to A whenever R is a relation from A to B.

(h) Prove:

$$\text{dom}(R^{-1}) = \text{ran}(R), \qquad \text{ran}(R^{-1}) = \text{dom}(R).$$

(i) Prove that $(R^{-1})^{-1} = R$.

5. Extra practice: working with relations

(j) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Consider the relation

$$R = \{(1, a), (2, a), (3, b)\}.$$

Find $\text{dom}(R)$ and $\text{ran}(R)$.

(k) Let $R = \{(1, 2), (2, 2), (3, 4)\}$ on $A = \{1, 2, 3, 4\}$. Compute R^{-1} and state its domain and range.

(l) Give an example of a relation on $A = \{1, 2, 3\}$ that has (i) empty domain, (ii) domain $\{1\}$, (iii) domain A.