Relations and Functions

Booklet 1: Sets, Subsets, Cartesian Products, and Relations

Mr. Merrick · December 8, 2025

Contents

Intro to Sets, Subsets, and Basic Notation

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Explainer

Goal. Build a foundation for all later work on relations and functions by learning the language of sets.

Key ideas.

- A **set** is a collection of distinct objects (elements).
- In **roster notation**, list elements explicitly:

$$A = \{1, 2, 3\}.$$

- Sets *ignore duplicates* and *ignore order.*
- $x \in A$ means x is an element of A.
- A **subset** $A \subseteq B$ means every element of A is in B.
- A **proper subset** $A \subset B$ means $A \subseteq B$ but $A \neq B$.

Worked example. Is $\{1, 2, 2, 3\} = \{3, 2, 1\}$? Yes — order and duplicates do not matter.

1. Identifying sets

(a) Which of the following represent valid sets? (i) $\{1, 2, 3\}$ (ii) $\{a, b, c, b\}$ (iii) $(1, 2, 3)$ (iv) $\{\text{cat}, \text{dog}, 7\}$

Solution. (i) Yes. (ii) Yes — duplicates are ignored, so it's really $\{a, b, c\}$. (iii) No — parentheses denote an ordered 3-tuple, not a set. (iv) Yes — sets can contain mixed types.

(b) List all elements of

$$A = \{3, 1, 4, 1, 5, 9, 2, 6\}.$$

Solution. Remove duplicates: $\{1, 2, 3, 4, 5, 6, 9\}$.

2. Membership and subsets

(c) Let $A = \{1, 3, 5, 7\}$. Determine whether:

i. $5 \in A$

ii. $2 \in A$

iii. $\{3, 7\} \subseteq A$

iv. $\{1, 2\} \subseteq A$

Solution. (i) True. (ii) False. (iii) True. (iv) False — 2 not in A.

(d) Compare $B = \{x, y, z\}$ and $C = \{z, x\}$. Determine which are true:

i. $C \subseteq B$

ii. $B \subseteq C$

iii. $C \subset B$

iv. $B = C$

Solution. (i) True. (ii) False. (iii) True — C is missing y. (iv) False.

3. Unusual elements of sets

(e) Let

$$X = \{\emptyset, 1, \{1\}\}.$$

Identify each element and answer: Is $\emptyset \in X$? Is $\emptyset \subseteq X$?

Solution. Elements: $\emptyset$, 1, and $\{1\}$. $\emptyset \in X$ is true. $\emptyset \subseteq X$ is always true (empty set is subset of every set).

(f) Compare

$$A = \{1, 2, \{3\}\}, \qquad B = \{1, 2, 3\}.$$

Are they equal?

Solution. No — $\{3\}$ is not the same object as 3.

4. Quick constructions

(g) Create a set with 5 elements: an integer, a decimal, a letter, a word, and a set.

Solution. Example: $\{7, 3.14, a, \text{“cat”}, \{1, 2\}\}$.

(h) Let

$$A = \{n \in \mathbb{Z} : -2 \le n \le 3\}.$$

Convert to roster notation.

Solution. $A = \{-2, -1, 0, 1, 2, 3\}$.

5. Subset counting

(i) How many subsets does $A = \{1, 2, 3, 4\}$ have?

Solution. $2^4 = 16$. Each element may be “in” or “out.”

6. Extra practice: sets and subsets

(j) Decide whether the following pairs of sets are equal. If not, explain why. (i) $\{1, 2, 3\}$ and $\{3, 2, 1, 1\}$ (ii) $\{a, \{b\}\}$ and $\{a, b\}$ (iii) $\{\emptyset\}$ and $\emptyset$

Solution. (i) Equal — order and duplicates do not matter. (ii) Not equal — in the first, $\{b\}$ is an element; in the second, b is an element. (iii) Not equal — $\emptyset$ has no elements; $\{\emptyset\}$ has one element, the empty set.

(k) Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$. List three different subsets of U that contain A as a subset.

Solution. Examples: $\{2, 4\}$, $\{1, 2, 4\}$, $\{2, 3, 4, 5\}$. Any supersets of A inside U are fine.

(l) Write each in roster notation: (i) $\{n \in \mathbb{Z} : 0 \le n \le 5\}$ (ii) $\{n \in \mathbb{Z} : -3 < n < 2\}$

Solution. (i) $\{0, 1, 2, 3, 4, 5\}$. (ii) $\{-2, -1, 0, 1\}$.

(m) Describe in set-builder notation the set

$$B = \{-5, -3, -1, 1, 3, 5\}.$$

Solution. $B = \{n \in \mathbb{Z} : n \text{ is odd and } -5 \le n \le 5\}$.

Set-Builder Notation, Empty Set, and Power Sets

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Explainer

Goal. Move between roster and set-builder notation and understand the empty set more formally.

Key ideas.

- Set-builder notation describes elements using a condition:
$$A = \{x \in \mathbb{Z} : x \text{ is odd}\}.$$
- The symbol ":" reads "such that."
- The **empty set** $\emptyset$ has no elements.
- The **power set** $\mathcal{P}(A)$ is the set of all subsets of A.

Worked example. $\{2, 4, 6, 8\} = \{n \in \mathbb{Z} : 2 \leq n \leq 8,\ n \text{ even}\}$.

1. Roster ↔ builder

(a) Convert to builder notation:
$$A = \{1, 3, 5, 7, 9\}.$$

Solution. $A = \{n \in \mathbb{Z} : n \text{ odd and } 1 \leq n \leq 9\}$.

(b) Convert to roster form:
$$B = \{n \in \mathbb{Z} : -3 \leq n \leq 2\}.$$

Solution. $B = \{-3, -2, -1, 0, 1, 2\}$.

(c) Fix the domain:
$$C = \{x : x > 0\}.$$

Solution. Should specify universe: $C = \{x \in \mathbb{R} : x > 0\}$.

2. Empty set basics

(d) Which describe $\emptyset$?

i. $\{x \in \mathbb{R} : x^2 = -1\}$

ii. $\{n \in \mathbb{Z} : n \text{ even and odd}\}$

iii. $\{x \in \mathbb{R} : x < 0 \text{ and } x > 10\}$

Solution. All three define the empty set.

(e) Is $\emptyset \subseteq \{\emptyset\}$? Is $\emptyset \in \{\emptyset\}$?

Solution. $\emptyset \subseteq \{\emptyset\}$ is true. $\emptyset \in \{\emptyset\}$ is also true (it contains the empty set as its element).

3. Power sets

(f) List all subsets of $A = \{x, y, z\}$.

Solution. $\emptyset$, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x, y, z\}$.

(g) How many subsets does a set of size n have?

Solution. 2^n subsets.

4. Why 2^n?

(h) Explain the reasoning behind $|\mathcal{P}(A)| = 2^{|A|}$.

Solution. Each element of A has two choices: in or out. For n elements, 2^n subsets.

5. Extra practice: builder notation and power sets

(i) Write the set in roster form:

$$D = \{n \in \mathbb{Z} : -2 \leq n < 3\}.$$

Solution. $D = \{-2, -1, 0, 1, 2\}$.

(j) Write in set-builder notation:

$$E = \{-4, -1, 2, 5, 8\}.$$

Solution. $E = \{n \in \mathbb{Z} : n \equiv 2 \pmod 3, -4 \leq n \leq 8\}$ or more descriptively: "integers from -4 to 8 that are 3 apart starting at -4."

(k) Let $B = \{1, 2\}$. Write $\mathcal{P}(B)$ and then $|\mathcal{P}(B)|$.

Solution. $\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $|\mathcal{P}(B)| = 4$.

(l) Give a set-builder description of the empty set that looks different from parts (i)–(iii) above.

Solution. Example: $\{n \in \mathbb{Z} : n^2 = 3\}$ or $\{x \in \mathbb{R} : x \text{ is a largest real number}\}$.

Set-Builder Notation vs. Roster Notation

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Explainer

Goal. Learn how to write sets in two different ways:

- **Roster notation:** list all elements explicitly,

$$A = \{1, 3, 5, 7\}.$$

- **Set-builder notation:** describe elements using a rule,

$$A = \{n \in \mathbb{Z} : n \text{ is an odd integer from 1 to 7}\}.$$

Key ideas.

- Roster is good for small, finite sets.
- Set-builder is good when:
  - the set is infinite,
  - the set follows a pattern,
  - the property is easier to describe than enumerate.
- The colon ":" can be read as "such that."
- Set-builder notation uses a membership requirement:

$$\{x \in (\text{universe}) : \text{condition on } x\}.$$

Worked example. Convert the set $A = \{2, 4, 6, 8\}$ into set-builder notation:

$$A = \{n \in \mathbb{Z} : n \text{ is even and } 2 \leq n \leq 8\}.$$

1. Converting roster → set-builder

(a) Convert the following set to set-builder notation:

$$A = \{1, 3, 5, 7, 9\}.$$

Solution. $A = \{n \in \mathbb{Z} : n \text{ is odd and } 1 \leq n \leq 9\}.$

(b) Write the set-builder form of:

$$B = \{-4, -2, 0, 2, 4\}.$$

Solution. $B = \{n \in \mathbb{Z} : -4 \leq n \leq 4 \text{ and } n \text{ is even}\}.$

(c) Convert to set-builder notation:

$$C = \{\text{red}, \text{blue}, \text{green}\}.$$

Solution. $C = \{x : x \in \{\text{red}, \text{blue}, \text{green}\}\}$. (Since there's no numeric rule, we simply state the allowed elements.)

2. Converting set-builder → roster

(d) Convert the set

$$D = \{n \in \mathbb{Z} : 0 < n < 6\}$$

into roster notation.

Solution. $D = \{1, 2, 3, 4, 5\}$.

(e) Convert

$$E = \{x \in \mathbb{R} : x^2 = 4\}.$$

Solution. Solve $x^2 = 4$ to get $x = \pm 2$, so $E = \{-2, 2\}$.

(f) Convert the following set:

$$F = \{n \in \mathbb{Z} : n \text{ is a multiple of 3 and } -10 \leq n \leq 10\}.$$

Solution. $F = \{-9, -6, -3, 0, 3, 6, 9\}$.

3. Identifying mistakes in set-builder notation

(g) Consider the notation

$$G = \{x : x > 0\}.$$

What is missing?

Solution. The universe is missing. Should be something like: $G = \{x \in \mathbb{R} : x > 0\}$.

(h) Identify the error in:

$$H = \{n \in \mathbb{Z} : n = \text{even}\}.$$

Solution. "$n = \text{even}$" is not a valid condition. Should be $H = \{n \in \mathbb{Z} : n \text{ is even}\}$ or $H = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z}, n = 2k\}$.

(i) Determine whether the following set is well-defined:

$$J = \{x \in \mathbb{R} : x \text{ is a big number}\}.$$

Solution. Not well-defined. "Big number" is subjective and not a mathematical property.

4. Mixed practice

(j) Write the set of all integers that are multiples of 4 (use builder notation).

Solution. $\{n \in \mathbb{Z} : \exists k \in \mathbb{Z},\, n = 4k\}$.

(k) Give the roster notation for the set:

$$K = \{n \in \mathbb{Z} : n \text{ is prime and } 1 < n < 20\}.$$

Solution. $K = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

(l) Convert the set into roster form:

$$L = \{x \in \mathbb{R} : x^2 - 9 = 0\}.$$

Solution. $x^2 - 9 = 0 \Rightarrow x = \pm 3$, so $L = \{-3, 3\}$.

(m) Convert to set-builder notation:

$$M = \{-5, -3, -1, 1, 3, 5\}.$$

Solution. $M = \{n \in \mathbb{Z} : n \text{ is odd and } -5 \leq n \leq 5\}$.

5. Extra practice: more conversions

(n) Convert to roster form:

$$S = \{n \in \mathbb{Z} : -1 \leq n \leq 4,\ n \text{ even}\}.$$

Solution. $S = \{0, 2, 4\}$.

(o) Convert to set-builder notation:

$$T = \{10, 20, 30, 40, \dots\}.$$

Solution. $T = \{10n : n \in \mathbb{N}\}$ or $T = \{n \in \mathbb{Z} : \exists k \in \mathbb{N},\, n = 10k\}$.

(p) Write a set-builder description of all real numbers between -3 and 5, including endpoints.

Solution. $\{x \in \mathbb{R} : -3 \leq x \leq 5\}$.

The Empty Set and Vacuous Truth

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Explainer

Goal. Deepen our understanding of $\emptyset$ and why statements about "all elements of $\emptyset$" are automatically true.

Key idea: Vacuous truth. A universal statement about the elements of $\emptyset$ is true because there are no counterexamples.
Example. "All unicorns are blue" is (logically) true — there are no unicorns to violate it.

1. Basic truth statements

(a) Determine whether each is true:

i. $\emptyset \in \emptyset$

ii. $\emptyset \subseteq \emptyset$

iii. $\{\emptyset\} \subseteq \emptyset$

Solution. (i) False. (ii) True (vacuously). (iii) False.

(b) Is the statement "Every element of $\emptyset$ is prime" true?

Solution. Yes — no counterexample exists.

2. Builder-notation emptiness

(c) Which sets are empty?

i. $\{x \in \mathbb{R} : x^2 = 4\}$

ii. $\{x \in \mathbb{R} : x^2 = 5\}$

iii. $\{x \in \mathbb{R} : x^2 = -9\}$

Solution. (i) Not empty ($x = \pm 2$). (ii) Not empty ($\pm\sqrt{5}$). (iii) Empty (no real square root of -9).

3. Universal statements over empty sets

(d) Explain why

$$(\forall x \in \emptyset)\ x > 1000$$

is true.

Solution. There are no elements in $\emptyset$ that could make the statement false.

(e) Provide an example of a false *existential* statement involving $\emptyset$.

Solution. "There exists $x \in \emptyset$ such that $x = 0$" is false.

4. Extra practice: truth with $\emptyset$

(f) Decide whether each statement is true or false. Briefly justify. (i) $\forall x \in \emptyset,\ x^2 > 0$ (ii) $\exists x \in \emptyset,\ x^2 = 1$ (iii) $\exists x \in \emptyset,\ x$ is an integer

Solution. (i) True (vacuously, there is no x to break the rule). (ii) False — there is no element at all. (iii) False for the same reason.

(g) Give a real-world "vacuously true" statement (like the unicorn example) and explain why it is vacuously true.

Solution. Example: "Every student in my class who is 200 years old can fly." There are no 200-year-old students, so there is no counterexample.

Cartesian Products and Grids

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Explainer

Goal. Understand ordered pairs and the structure of $A \times B$ as the foundation for defining relations and functions.

Key ideas.

- $A \times B = \{(a, b) : a \in A,\ b \in B\}$.
- Order matters: $(a, b) \neq (b, a)$ in general.
- $|A \times B| = |A| \cdot |B|$.
- If A or B is empty, $A \times B = \emptyset$.

1. Listing and counting

(a) Let $A = \{1, 2\}$ and $B = \{x, y, z\}$.

i. List $A \times B$.

Solution. $\{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$.

ii. How many elements does $A \times B$ have?

Solution. $2 \cdot 3 = 6$.

(b) Without listing, compute $|P \times Q|$ if $|P| = 5$ and $|Q| = 7$.

Solution. $5 \cdot 7 = 35$.

2. Empty-set cases

(c) Compute $A \times \emptyset$.

Solution. $\emptyset$.

(d) Compute $\emptyset \times B$.

Solution. $\emptyset$.

3. Grid interpretation

(e) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Draw or describe $A \times B$ as a table or grid.

Solution. Rows = elements of A, columns = elements of B, each cell is (a, b). So the grid entries are $(1, a), (1, b)$ in row 1; $(2, a), (2, b)$ in row 2; $(3, a), (3, b)$ in row 3.

4. Product proofs

(f) Prove: If $A \subseteq B$ then $A \times C \subseteq B \times C$.

Solution. Let $(a, c) \in A \times C$. Then $a \in A \subseteq B$, so $(a, c) \in B \times C$.

(g) Provide a counterexample showing the converse need not hold.

Solution. Let $C = \emptyset$, $A = \{1\}$, $B = \emptyset$. Then $A \times C = B \times C = \emptyset$, but $A \not\subseteq B$.

(h) Prove:

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Solution. Standard double-inclusion proof. Both sides consist of pairs (a, x) with $a \in A$ and x in both B and C.

5. Extra practice: more products

(i) Let $A = \{0, 1, 2\}$ and $B = \{p, q\}$. List all elements of $A \times B$ and of $B \times A$. Are they the same set?

Solution. $A \times B = \{(0, p), (0, q), (1, p), (1, q), (2, p), (2, q)\}$. $B \times A = \{(p, 0), (p, 1), (p, 2), (q, 0), (q, 1), (q, 2)\}$. They are *not* the same: the order of coordinates is different.

(j) If $|A| = 3$, $|B| = 4$, and $|C| = 2$, compute $|A \times B \times C|$.

Solution. $3 \cdot 4 \cdot 2 = 24$.

(k) Suppose A has 5 elements. How many elements does $A \times A$ have? What about $A \times A \times A$?

Solution. $|A \times A| = 5 \cdot 5 = 25$. $|A \times A \times A| = 5^3 = 125$.

Relations as Subsets of Cartesian Products

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Explainer

Goal. Understand relations as sets of ordered pairs taken from a Cartesian product. This prepares us to define functions as *special* relations.

Key ideas.

- A relation R from A to B is any subset of $A \times B$:

$$R \subseteq A \times B.$$

- The **domain** of R:

$$\mathrm{dom}(R) = \{a \in A : \exists b,\ (a,b) \in R\}.$$

- The **range** of R:

$$\mathrm{ran}(R) = \{b \in B : \exists a,\ (a,b) \in R\}.$$

- The **inverse relation**:

$$R^{-1} = \{(b,a) : (a,b) \in R\}.$$

Worked example. If $R = \{(1,a),(3,b)\}$ from $\{1,2,3\}$ to $\{a,b\}$ then $\mathrm{dom}(R) = \{1,3\}$ and $\mathrm{ran}(R) = \{a,b\}$.

1. Identifying relations

(a) Let $A = \{1,2\}$, $B = \{x,y,z\}$ and

$$R = \{(1,y),(2,x)\}.$$

Is $R \subseteq A \times B$?

Solution. Yes — each pair has first element in A and second in B.

(b) Let

$$S = \{(x,1),(2,1)\}, \qquad A = \{x,y\}, \quad B = \{1,2,3\}.$$

Is S a relation from A to B?

Solution. No — the first coordinate 2 is not in A.

2. Domain and range

(c) Let

$$R = \{(1, a), (1, b), (2, a), (3, c)\}.$$

Find $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$.

Solution. Domain: $\{1, 2, 3\}$. Range: $\{a, b, c\}$.

(d) Let

$$T = \{(m, n) \in \mathbb{Z}^2 : m < n\}.$$

Describe the domain and range.

Solution. Domain: all integers $\mathbb{Z}$; every m has some $n > m$. Range: all integers $\mathbb{Z}$; every n has some $m < n$.

3. Inverse relations

(e) Let

$$R = \{(1, a), (2, b), (3, b)\}.$$

Compute R^{-1}.

Solution. $R^{-1} = \{(a, 1), (b, 2), (b, 3)\}$.

(f) Let

$$D = \{(m, n) \in \mathbb{Z}^2 : m \mid n\}.$$

Describe D^{-1} in words.

Solution. D^{-1} contains all pairs (n, m) such that m divides n. In words: "the first number is divisible by the second."

4. Relation proofs

(g) Prove that R^{-1} is a relation from B to A whenever R is a relation from A to B.

Solution. If $(b, a) \in R^{-1}$, then $(a, b) \in R$. So $a \in A$ and $b \in B$, hence $(b, a) \in B \times A$. Therefore $R^{-1} \subseteq B \times A$.

(h) Prove:

$$\mathrm{dom}(R^{-1}) = \mathrm{ran}(R), \qquad \mathrm{ran}(R^{-1}) = \mathrm{dom}(R).$$

Solution. $(b, a) \in R^{-1}$ iff $(a, b) \in R$. So b appears as a first coordinate of R^{-1} exactly when it appears as a second coordinate of R. Likewise for range.

(i) Prove that $(R^{-1})^{-1} = R$.

Solution. By definition: $(b, a) \in R^{-1}$ iff $(a, b) \in R$. Flipping again returns each pair to its original position.

5. Extra practice: working with relations

(j) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Consider the relation

$$R = \{(1, a), (2, a), (3, b)\}.$$

Find $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$.

Solution. $\mathrm{dom}(R) = \{1, 2, 3\}$, $\mathrm{ran}(R) = \{a, b\}$.

(k) Let $R = \{(1, 2), (2, 2), (3, 4)\}$ on $A = \{1, 2, 3, 4\}$. Compute R^{-1} and state its domain and range.

Solution. $R^{-1} = \{(2, 1), (2, 2), (4, 3)\}$. Domain $= \{2, 4\}$, range $= \{1, 2, 3\}$.

(l) Give an example of a relation on $A = \{1, 2, 3\}$ that has (i) empty domain, (ii) domain $\{1\}$, (iii) domain A.

Solution. (i) $R = \emptyset$. (ii) $R = \{(1, 1)\}$. (iii) $R = \{(1, 1), (2, 1), (3, 1)\}$, for example.