

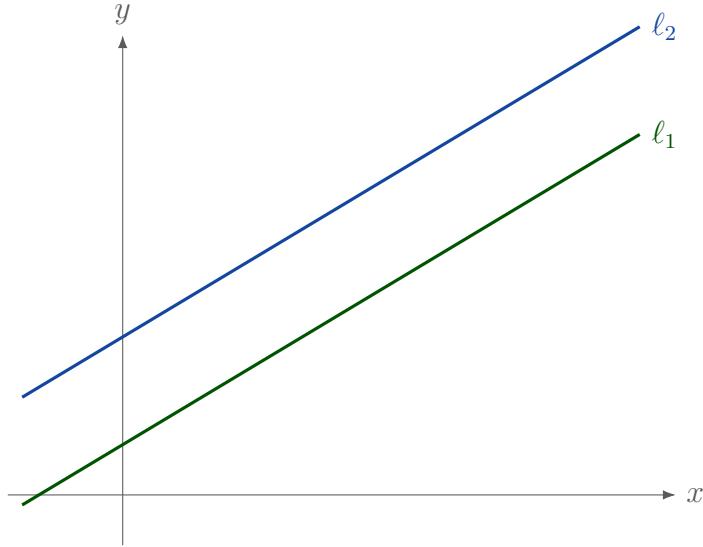
# DISTANCE BETWEEN LINES & SHOELACE AREA

Mr. Merrick · Math 10 · January 20, 2026

## 1) Distance Between Two Parallel Lines

Consider

$$\ell_1 : y = \frac{3}{5}x + \frac{7}{10} \quad \ell_2 : y = \frac{3}{5}x + \frac{11}{5}.$$



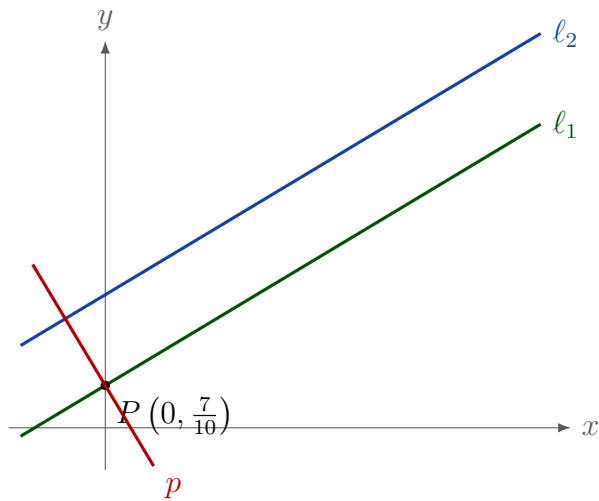
**Add a perpendicular line.**

Both lines have slope  $m = \frac{3}{5}$ , so a perpendicular slope is

$$m_{\perp} = -\frac{1}{m} = -\frac{5}{3}.$$

Take an easy point  $P = (0, \frac{7}{10})$ , the  $y$ -intercept of  $\ell_1$ . The line perpendicular to  $\ell_1$  through  $P$  is:

$$p : y = -\frac{5}{3}x + \frac{7}{10}.$$



Solve for the intersection point  $Q$  (where  $p$  meets  $\ell_2$ ).

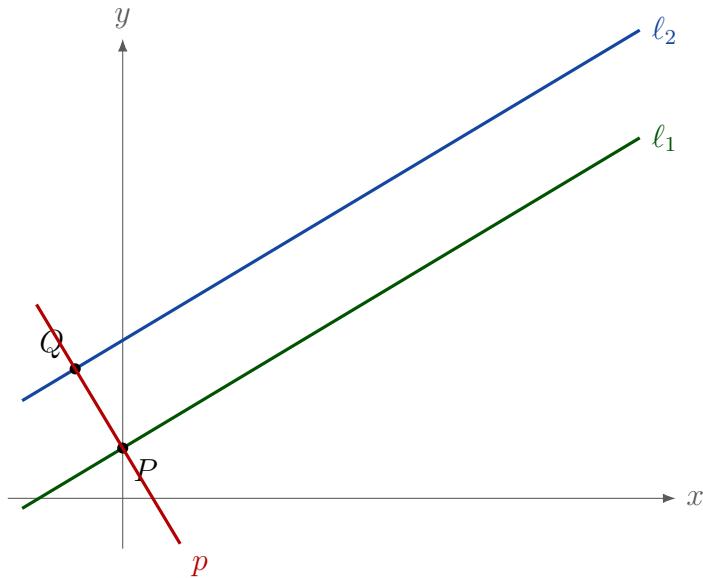
$$\frac{3}{5}x + \frac{11}{5} = -\frac{5}{3}x + \frac{7}{10} \Rightarrow \left(\frac{3}{5} + \frac{5}{3}\right)x = \frac{7}{10} - \frac{11}{5} \Rightarrow \frac{34}{15}x = -\frac{3}{2} \Rightarrow x = -\frac{45}{68}.$$

Then

$$y = \frac{3}{5}x + \frac{11}{5} = \frac{3}{5}\left(-\frac{45}{68}\right) + \frac{11}{5} = \frac{11}{5} - \frac{27}{68}.$$

So

$$Q\left(-\frac{45}{68}, \frac{11}{5} - \frac{27}{68}\right).$$



The distance is the length of the perpendicular segment  $PQ$ .

$$d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{\left(-\frac{45}{68}\right)^2 + \left(\left(\frac{11}{5} - \frac{27}{68}\right) - \frac{7}{10}\right)^2}.$$

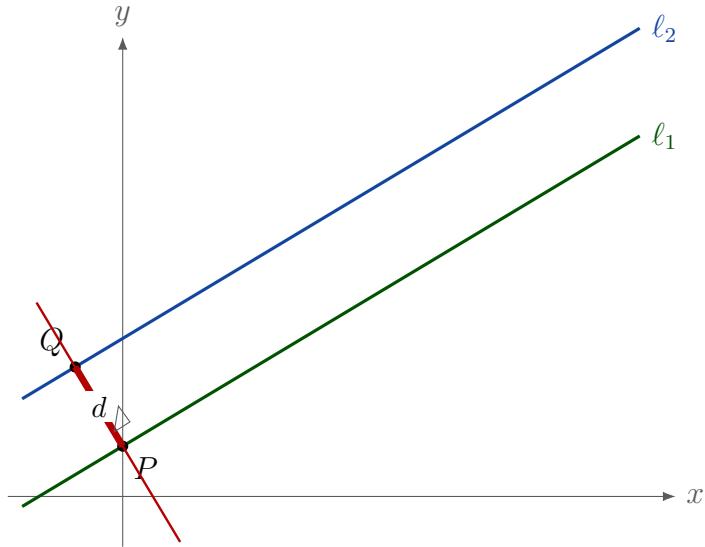
But

$$\left(\frac{11}{5} - \frac{27}{68}\right) - \frac{7}{10} = \frac{3}{2} - \frac{27}{68} = \frac{75}{68},$$

so

$$d = \sqrt{\left(\frac{45}{68}\right)^2 + \left(\frac{75}{68}\right)^2} = \frac{1}{68}\sqrt{45^2 + 75^2} = \frac{15\sqrt{34}}{68} \approx 1.29.$$

$$d = \frac{15\sqrt{34}}{68} \approx 1.29$$



### The general distance formula

If two parallel lines are written in general form,

$$Ax + By + C_1 = 0 \quad Ax + By + C_2 = 0,$$

their distance is

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

### Why this formula works (proof)

Assume  $B \neq 0$  so each line can be written as  $y = mx + b$ . (If  $B = 0$ , the lines are vertical, and the same final formula still applies.)

Same steps as the example: rewrite as  $y = mx + b$ , draw a perpendicular through an easy point, intersect, then use the distance formula. Carrying it out with symbols gives

$$d = \frac{|b_1 - b_2|}{\sqrt{1 + m^2}} \quad \Rightarrow \quad d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

## 2) Shoelace Formula (Area of a Triangle)

Find the area of the triangle formed by the three points

Let

$$A(1, 1), \quad B(6, 2), \quad C(3, 6).$$

Write them in order and repeat the first point:

$x$	$y$
1	1
6	2
3	6
1	1

Down-right products:

$$(1)(2) + (6)(6) + (3)(1) = 41.$$

Up-right products:

$$(1)(6) + (2)(3) + (6)(1) = 18.$$

So

$$\text{Area} = \frac{1}{2}|41 - 18| = 11.5.$$

$$\boxed{\text{Area} = 11.5}$$

### The general shoelace formula

For  $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ ,

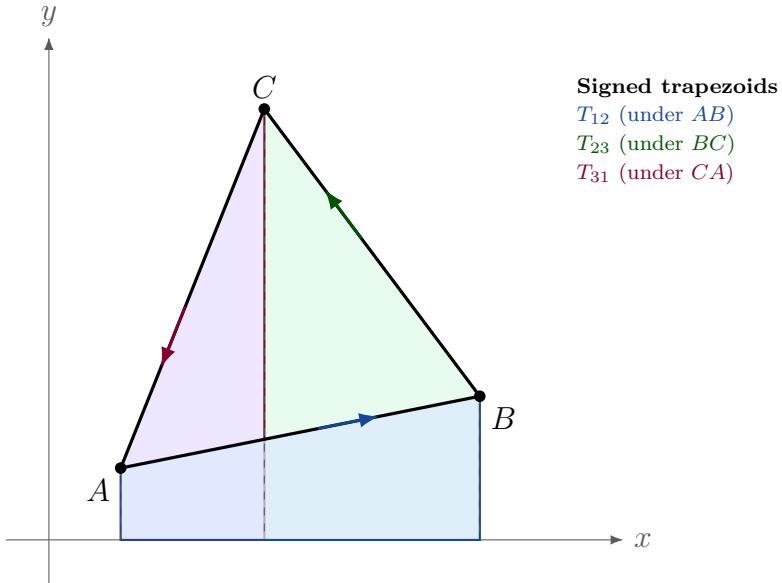
$$\boxed{\text{Area} = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - (x_2y_1 + x_3y_2 + x_1y_3)|}$$

This idea extends to polygons with more than three sides. More information can be found here:

[https://artofproblemsolving.com/wiki/index.php/Shoelace\\_Theorem](https://artofproblemsolving.com/wiki/index.php/Shoelace_Theorem)

## Why the shoelace formula works (proof idea)

Add the signed trapezoid areas between each side of the triangle and the  $x$ -axis.



Each side of the triangle, together with the  $x$ -axis, forms a trapezoid. For the side from  $A(x_1, y_1)$  to  $B(x_2, y_2)$ , the parallel sides have lengths  $y_1$  and  $y_2$  and the width is  $(x_2 - x_1)$ , so

$$T_{12} = \frac{1}{2}(y_1 + y_2)(x_2 - x_1).$$

Similarly,

$$T_{23} = \frac{1}{2}(y_2 + y_3)(x_3 - x_2), \quad T_{31} = \frac{1}{2}(y_3 + y_1)(x_1 - x_3).$$

Now sum the areas of the trapezoids:

$$T_{12} + T_{23} + T_{31}.$$

Because the triangle is traced in a consistent (clockwise) direction, regions outside the triangle appear once with a positive sign and once with a negative sign, so they cancel.

After cancellation, the only region counted exactly once is the interior of the triangle. Thus,  $T_{12} + T_{23} + T_{31}$  equals the *signed area* of the triangle. Expanding the sum gives

$$\begin{aligned} T_{12} + T_{23} + T_{31} &= \frac{1}{2} \left[ (y_1 + y_2)(x_2 - x_1) + (y_2 + y_3)(x_3 - x_2) + (y_3 + y_1)(x_1 - x_3) \right] \\ &= \frac{1}{2} \left( x_1 y_2 + x_2 y_3 + x_3 y_1 - (x_2 y_1 + x_3 y_2 + x_1 y_3) \right). \end{aligned}$$

Therefore, the area of the triangle is

$$\text{Area} = \frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - (x_2 y_1 + x_3 y_2 + x_1 y_3)|.$$